GENERALIZED LIMITS IN GENERAL ANALYSIS*

SECOND PAPER

BY

CHARLES N. MOORE

In a previous paper of the same title† I have developed the fundamental principles of a general theory which includes as particular instances the theories of Cesàro and Hölder summability of divergent series and divergent integrals. I further made use of these fundamental principles to prove a general theorem which includes as special cases several important theorems in the above mentioned special theories.

In the present paper the general theory referred to above is extended to the case of multiple limits and the theorem mentioned is likewise generalized. The theorem thus obtained includes as special cases the extension to multiple series of the Knopp-Schnee-Ford theorem‡ on the equivalence of Cesàro and Hölder summability for divergent series, the extension to multiple integrals of the analogous theorem of Landau‡ for the case of divergent integrals, and the extension to partial derivatives of a corresponding theorem with regard to the equivalence of certain generalized derivatives. Once the principles of the theory are set forth, the proof of this general theorem is fully as simple as the proofs of any of the special theorems would be. Thus we have exhibited the greater power of the methods of General Analysis as compared with the methods of classical analysis.

The basis of our general theory may be indicated as follows:

$$(\mathfrak{A};\ \mathfrak{P}_1;\ \mathfrak{P}_2;\ \ldots;\ \mathfrak{P}_m;\ \mathfrak{S}_1;\ \mathfrak{S}_2;\ \ldots;\ \mathfrak{S}_m;\ \mathfrak{G}^{\text{on } \mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_m, \text{ to } \mathfrak{A}};$$

$$\mathfrak{H}_i^{\text{on } \mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_m \text{ to } \mathfrak{A}} \qquad (i = 1, 2, \ldots, m);$$

$$\mathfrak{F}_i^{\text{on } \mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_m \text{ to } \mathfrak{A}} \qquad (i = 1, 2, \ldots, m);$$

$$\mathfrak{H}^{\text{on } \mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_m \text{ to } \mathfrak{A}};\ \mathfrak{F}^{\text{on } \mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_m \text{ to } \mathfrak{A}};\ \varphi_0^{(m)};$$

$$J_i^{\text{on } \mathfrak{G} \text{ to } \mathfrak{H}_i;\ \text{on } \mathfrak{H}_i \text{ to } \mathfrak{F}_i} \qquad (i = 1, 2, \ldots, m);$$

$$J^{\text{on } \mathfrak{G} \text{ to } \mathfrak{H};\ \text{on } \mathfrak{H} \text{ to } \mathfrak{F}}),$$

* Presented to the Society April 14, 1922.

† These Transactions, vol. 24 (1922), pp. 79-88.

‡ For references to the literature dealing with the special theorems referred to, see Paper I.

where $\mathfrak{A} \equiv [a]$ denotes the class of all real numbers a, $\mathfrak{P}_i \equiv [p_i]$ denotes a class of elements p_i $(i = 1, 2, \ldots, m)$, and $\mathfrak{S}_i \equiv [\sigma^{(i)}]$ denotes a class of sets $\sigma^{(i)}$ of elements p_i of the range $\mathfrak{P}_i$ $(i = 1, 2, \ldots, m)$; $\mathfrak{G} \equiv [\gamma]$, $\mathfrak{H}_i \equiv [\eta^{(i)}]$ $(i = 1, 2, \ldots, m)$, $\mathfrak{F}_i \equiv [\varphi^{(i)}]$ $(i = 1, 2, \ldots, m)$, $\mathfrak{H} \equiv [\eta]$, and $\mathfrak{F} \equiv [\varphi]$ are $(2m + 3)$ classes of functions, γ, $\eta^{(1)}, \ldots, \eta^{(m)}$, $\varphi^{(1)}, \ldots, \varphi^{(m)}$, η, and φ respectively on $\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_m$ to $\mathfrak{A}$ (we consider only single-valued functions); $\varphi_0^{(m)}$ is a special function of the class $\mathfrak{F}$; J_i is a functional operation turning a function of the class $\mathfrak{G}$ into a function of the class $\mathfrak{H}_i$ or a function of the class $\mathfrak{H}_i$ into a function of the class $\mathfrak{F}_i$, denoted by $J_i\gamma$ or $J_i\eta^{(i)}$ respectively; and J is a functional operation turning a function of the class $\mathfrak{G}$ into a function of the class $\mathfrak{H}$ or a function of the class $\mathfrak{H}$ into a function of the class $\mathfrak{F}$, denoted by $J\gamma$ or $J\eta$ respectively.

In order to show the relationship of our general theorem to the special cases of it to which we have referred, we will indicate here what the general basis reduces to in the particular instances III and IV.

$$\mathfrak{P}_i^{\mathrm{III}} \equiv [\text{all } n_i = 1, 2, 3, \ldots] \qquad (i = 1, 2, \ldots, m);$$

$$\mathfrak{S}_i \equiv [\sigma_{n_i} \equiv (1, 2, \ldots, n_i) | n_i] \qquad (i = 1, 2, \ldots, m);$$

$$\mathfrak{G} \equiv \mathfrak{H}_i \equiv \mathfrak{F}_i \equiv \mathfrak{H} \equiv \mathfrak{F} \equiv [\text{all } \gamma, \eta^{(i)}, \varphi^{(i)}, \eta, \varphi^{\text{on } \mathfrak{S}_1, \ldots, \mathfrak{S}_m \text{ to } \mathfrak{A}}] \qquad (i = 1, 2, \ldots, m);$$

$$\varphi_0^{(m)}(\sigma_{n_1}, \sigma_{n_2}, \ldots, \sigma_{n_m}) = n_1 n_2 \cdots n_m \qquad (n_i;\ i = 1, 2, \ldots, m);$$

$$(J_i\,\theta)(\sigma_{n_1}, \ldots, \sigma_{n_m}) = \sum_{k_i=1}^{k_i=n_i} \theta(\sigma_{n_1}, \ldots, \sigma_{k_i}, \ldots, \sigma_{n_m}) \qquad (n_i;\ i = 1, 2, \ldots, m;\ \theta = \gamma, \eta^{(i)});$$

$$(J\theta)(\sigma_{n_1}, \ldots, \sigma_{n_m}) = \sum_{k_1=1}^{k_1=n_1} \cdots \sum_{k_m=1}^{k_m=n_m} \theta(\sigma_{k_1}, \ldots, \sigma_{k_m}) \qquad (n_i;\ i = 1, 2, \ldots, m;\ \theta = \gamma, \eta).$$

$$\mathfrak{P}_i^{\mathrm{IV}} \equiv [\text{all } a_i > 0] \qquad (i = 1, 2, \ldots, m);$$

$$\mathfrak{S}_i \equiv [\sigma_a^{(i)} \equiv (\text{all } x_i \text{ such that } 0 < x_i \leq a_i) \quad (a_i > 0;\ i = 1, 2, \ldots, m)];$$

$\mathfrak{G} \equiv$ [all functions that are finite in any finite region $(0 < x_i \leq a_i;$ $i = 1, 2, \ldots, m)$ and are integrable (Lebesgue) with respect

to each of the variables x_i ($i = 1, \ldots, m$) on every finite interval ($0 < x_i \leqq a_i$; $i = 1, \ldots, m$)];

$$\mathfrak{H}_i \equiv \text{all } \eta_i^{(i)} = \int_0^{x_i} \gamma(x_1, x_2, \ldots, x_i, \ldots, x_m)\,dx_i \quad (x_i > 0;\ i = 1, \ldots, m);$$

$$\mathfrak{F}_i \equiv \text{all } \varphi^{(i)} = \int_0^{x_i} \eta^{(i)}(x_1, x_2, \ldots, x_i, \ldots, x_m)\,dx_i \quad (x_i > 0;\ i = 1, \ldots, m);$$

$$\mathfrak{H} \equiv \text{all } \eta = \int_0^{x_1} \cdots \int_0^{x_m} \gamma(x_1, \ldots, x_m) \quad (x_i > 0;\ i = 1, \ldots, m);$$

$$\mathfrak{F} \equiv \text{all } \varphi = \int_0^{x_1} \cdots \int_0^{x_m} \eta(x_1, \ldots, x_m) \quad (x_i > 0;\ i = 1, \ldots, m);$$

$$(\varphi_0^{(m)})(\sigma_{a_1}, \ldots, \sigma_{a_m}) = a_1\, a_2 \cdots a_m \quad (a_i;\ i = 1, \ldots, m);$$

$$(J_i\theta)(\sigma_{a_1}, \ldots, \sigma_{a_m}) = \int_0^{a_i} \theta\,dx_i \quad (a_i;\ i = 1, \ldots, m;\ \theta = \gamma, \eta^{(i)});$$

$$(J\theta)(\sigma_{a_1}, \ldots, \sigma_{a_m}) = \int_0^{a_1} \cdots \int_0^{a_m} \theta \quad (a_i;\ i = 1, \ldots, m;\ \theta = \gamma, \eta).$$

With regard to each of the classes $\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_m$ we make definitions analogous to those made for the class $\mathfrak{S}$ in Paper I, and we further postulate analogous properties. These properties will be referred to by the same letters as in the previous paper with a subscript or index attached to indicate the particular class to which reference is made. When any two functions θ, α on $\mathfrak{S}_1, \ldots, \mathfrak{S}_m$ to $\mathfrak{A}$ are regarded as functions of a single set $\sigma^{(i)}$, the other sets being held fixed, we define the notation $(D_i\theta)(\sigma^{(1)}, \ldots, \sigma^{(m)}) = \alpha(\sigma^{(1)}, \ldots, \sigma^{(m)})$

in a manner entirely analogous to that in which the notation $(D\theta)(\sigma) = \alpha(\sigma)$ was defined in Paper I.

When functions of the class $\mathfrak{G}$ are regarded as functions of a single set $\sigma^{(i)}$, the other sets being held fixed, we postulate for them the properties of class $\mathfrak{G}$ of Paper I, these properties to be referred to by the same letter with suitable index or subscript. Analogous properties for the classes $\mathfrak{H}$ and $\mathfrak{F}$, designated in similar fashion, are also postulated. Furthermore for the classes $\mathfrak{H}_i$ and $\mathfrak{F}_i$, regarded as functions of the set $\sigma^{(i)}$ alone, we postulate the properties of classes $\mathfrak{H}$ and $\mathfrak{F}$ of Paper I and indicate them in like manner. When the functions of classes $\mathfrak{G}$ and $\mathfrak{H}_i$ that are involved in the operation J_i are regarded as functions of the set $\sigma^{(i)}$ alone, we require J_i to have all the properties required of J in Paper I, which properties we shall designate by the same symbols with subscript or index i. We further postulate that any of the operations $J_1, J_2, \ldots, J_m$ is interchangeable with any other of the set, which property we designate as (I). We also postulate as to the relationship between J and $J_1, J_2, \ldots, J_m$ that

$$(N) \qquad (J\eta)(\sigma^{(1)}, \ldots, \sigma^{(m)}) \equiv (J_1(J_2 \cdots (J_m \eta) \cdots))(\sigma^{(1)}, \ldots, \sigma^{(m)}).$$

With regard to the special function $\varphi_0^{(m)}(\sigma^{(1)}, \ldots, \sigma^{(m)})$ we postulate that

$$(X) \qquad \varphi_0^{(m)}(\sigma^{(1)}, \ldots, \sigma^{(m)}) = \varphi_0(\sigma^{(1)})\,\varphi_0(\sigma^{(2)}) \cdots \varphi_0(\sigma^{(m)}),$$

where $\varphi_0(\sigma^{(i)})$ as function of $\sigma^{(i)}$, for $i = 1, 2, \ldots, m$, is the same function as $\varphi_0(\sigma)$ of Paper I as function of σ. We also postulate for the class $\mathfrak{F}$ that $(J_i\varphi)(\sigma^{(1)}, \ldots, \sigma^{(m)})$, for $i = 1, 2, \ldots, m$, is of the class $\mathfrak{F}$, which property we designate as (K).

For the sake of brevity we shall agree to represent, in all cases where no loss of clearness is involved, the set of elements $p^{(1)}, \ldots, p^{(m)}$ by the single symbol p, the set of classes $\mathfrak{P}^{(1)}, \ldots, \mathfrak{P}^{(m)}$ by $\mathfrak{P}$, the set of classes $\mathfrak{S}^{(1)}, \ldots, \mathfrak{S}^{(m)}$ by $\mathfrak{S}$, and the set of sets $\sigma^{(1)}, \ldots, \sigma^{(m)}$ by σ. Analogous to the definition of the notation $\lim_\sigma \theta(\sigma) = a$ in Paper I, we define the corresponding notation in the case that σ represents the set of sets $\sigma^{(1)}, \ldots, \sigma^{(m)}$ to mean that corresponding to every positive e there exist sets $\sigma_e^{(1)}, \ldots, \sigma_e^{(m)}$ such that for sets $\sigma^{(i)} > \sigma_e^{(i)}$ $(i = 1, 2, \ldots, m)$ we have $|\theta(\sigma) - a| < e$.

We then postulate as to the class $\mathfrak{H}$ the property (B) defined by

(B) If $\lim_\sigma \eta(\sigma)$ exists and is equal to a, then $|\eta(\sigma)| < a_1\ (\sigma)$.

If we also for the sake of brevity agree to represent a group of properties such as $R_1, R_2, \ldots, R_m$ by the single letter R, we may indicate the foundation of our theory as follows:

$$\Sigma = \Big(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{S}^{UAR};\ \mathfrak{G}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}\,.\,LP};\ \mathfrak{H}_i^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}\,.\,L_i P_i S_G^{(i)}} \qquad (i = 1, \ldots, m);$$

$$\mathfrak{F}_i^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}\,.\,L_i P_i S_H^{(i)} C_i \Delta_i} \qquad (i = 1, \ldots, m); \qquad \mathfrak{H}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}\,.\,LPS_G B};$$

$$\mathfrak{F}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}\,.\,LPS_H C \Delta K}; \qquad \varphi_0^{(m)\mathfrak{F}\,.\,X};$$

$$J_i^{\text{on } \mathfrak{G} \text{ to } \mathfrak{H}_i\,.\,\text{on } \mathfrak{H}_i \text{ to } \mathfrak{F}_i\,.\,M_1^{(i)} M_2^{(i)} I_D^{(i)} I_J^{(i)} I} \qquad (i = 1, \ldots, m);$$

$$J^{\text{on } \mathfrak{G} \text{ to } \mathfrak{H}\,.\,\text{on } \mathfrak{H} \text{ to } \mathfrak{F}\,.\,N}\Big).$$

We now set

$$\varphi_{0n}^{(m)}(\sigma) = \varphi_{0n}(\sigma^{(1)})\,\varphi_{0n}(\sigma^{(2)}) \cdots \varphi_{0n}(\sigma^{(m)}), \tag{1}$$

where $\varphi_{0n}(\sigma^{(i)})$, as function of $\sigma^{(i)}$, is the same function as $\varphi_{0n}(\sigma)$, as function of σ, defined by equation (3) of Paper I. We are then ready to define the two generalized limits with which we shall be concerned. Given any function $\eta(\sigma)$, we set

$$(C_n \eta)(\sigma) = \frac{(n!)^m}{\varphi_{0n}^{(m)}(\sigma)} (J^n \eta)(\sigma) \qquad (n), \tag{2}$$

$$(M \eta)(\sigma) = \frac{1}{\varphi_0^{(m)}(\sigma)} (J\eta)(\sigma), \tag{3}$$

$$(H_n \eta)(\sigma) = (M^n \eta)(\sigma) \qquad (n). \tag{4}$$

If for a fixed n $\lim_\sigma (C_n \eta)(\sigma)$ exists, we define this limit as the generalized limit of type (Cn) for $\eta(\sigma)$. If $\lim_\sigma (H_n \eta)(\sigma)$ exists, we define this limit as the generalized limit of type (Hn) for $\eta(\sigma)$.

Before proceeding to the proof of the equivalence theorem we introduce the following notations:

$$(5)\qquad (M_i\gamma)(\sigma) = [1/\varphi_0(\sigma^{(i)})](J_i\gamma)(\sigma) \qquad (i=1,\ldots,m),$$

$$(6)\quad (C_n^{i,i+1,\ldots,i+k}\eta)(\sigma) = \frac{(n!)^{k+1}}{\varphi_{0n}(\sigma^{(i)})\cdots\varphi_{0n}(\sigma^{(i+k)})}(J_i\cdots(J_{i+k}\eta)\cdots)(\sigma) \quad (n),$$

$$(7)\quad \begin{aligned} &\gamma_n^{(i)}(\sigma) = \varphi_0(\sigma^{(i)})\,\varphi_0(\sigma_1^{(i)})\cdots\varphi_0(\sigma_{n-2}^{(i)})\,\gamma(\sigma) \qquad (n>2;\ i=1,\ldots,m),\\ &\gamma_2^{(i)}(\sigma) = \varphi_0(\sigma^{(i)})\,\gamma(\sigma), \qquad \gamma_1^{(i)}(\sigma) = \gamma(\sigma) \qquad (i=1,\ldots,m),\end{aligned}$$

where $\sigma_1^{(i)}, \sigma_2^{(i)}, \ldots$ are defined with regard to $\sigma^{(i)}$ in the same manner as $\sigma_1, \sigma_2, \ldots$ with regard to σ in Paper I;

$$(8)\quad (S_n^{(i)}\gamma)(\sigma) = \left(\left(\frac{n-1}{n}M_i + \frac{1}{n}E\right)\gamma\right)(\sigma) \qquad (n),$$

$$(9)\quad (S_n^{(1,2,\ldots,i)}\gamma)(\sigma) = (S_n^{(1)}(S_n^{(2)}\cdots(S_n^{(i)}\gamma)\cdots))(\sigma) \quad (n;\ i=1,\ldots,m),$$

$$(10)\quad (S_n\gamma)(\sigma) = (S_n^{(1)}(S_n^{(2)}\cdots(S_n^{(m)}\gamma)\cdots))(\sigma) \qquad (n),$$

$$(11)\quad (T_n^{(i)}\gamma)(\sigma) = n\gamma(\sigma) - \frac{n(n-1)}{\varphi_{0n}(\sigma^{(i)})}(J_i\gamma_n^{(i)})(\sigma) \qquad (n;\ i=1,\ldots,m).$$

We are now ready for the proof of our theorem; we begin by proving some lemmas.

LEMMA 1. *If we define S_n as in* (10), *we have the identity*

$$(12)\qquad (S_n(C_n\eta))(\sigma) = (M(C_{n-1}\eta))(\sigma) \qquad (n).$$

We have from Lemma 1 of Paper I and the interchangeability of the various operations involved

$$
\begin{aligned}
(S_n(C_n\,\eta))(\sigma) &= \Big(S_n^{(1,2,\ldots,m-1)}\,(C_n^{(1,2,\ldots,m-1)}\,(S_n^{(m)}\,(C_n^{(m)}\,\eta)))\Big)(\sigma)\\
&= \Big(S_n^{(1,2,\ldots,m-1)}\,(C_n^{(1,2,\ldots,m-1)}\,(M_m\,(C_{n-1}^{(m)}\,\eta)))\Big)(\sigma)\\
&= \Big(S_n^{(1,2,\ldots,m-2)}\,\Big(C_n^{(1,2,\ldots,m-2)}\,\Big(S_n^{(m-1)}\,(C_n^{(m-1)}\,(M_m\,(C_{n-1}^{(m)}\,\eta)))\Big)\Big)\Big)(\sigma)\\
&= \Big(S_n^{(1,2,\ldots,m-2)}\,\Big(C_n^{(1,2,\ldots,m-2)}\,(M_{m-1}\,(M_m\,(C_{n-1}^{(m-1,m)}\,\eta)))\Big)\Big)(\sigma)\\
&= \;.\\
&= \Big(M_1\,(M_2\,\cdots\,(M_n\,(C_{n-1}\,\eta))\,\cdots\,)\Big)(\sigma) = (M(C_{n-1}\,\eta))(\sigma).
\end{aligned}
$$

Our lemma is therefore established.

We define $\varphi_n^{(i)}$ in a manner analogous to the definition of $\gamma_n^{(i)}$ in (7). We also set

$$\varphi_{0n\cdot i}(\sigma) = \varphi_{0n}(\sigma^{(i)}).$$

We then prove

LEMMA 2. *If* $\lim_\sigma \varphi(\sigma)$ *exists and is equal to* a *and* $|\varphi(\sigma)| < a_1$ *for every* σ, *then* $\lim_\sigma [\varphi_{0n}(\sigma^{(i)})]^{-1}(J_i\,\varphi_n^{(i)})(\sigma)$ *will exist and be equal to* a/n *and we shall have*

$$|[\varphi_{0n}(\sigma^{(i)})]^{-1}(J_i\,\varphi_n^{(i)})(\sigma)| < a_1^{(i)} \qquad (\sigma,\, n;\; i = 1, \ldots, m).$$

Given a positive e, we choose σ_e' so that $a - (e/4) < \varphi(\sigma) < a + (e/4)$ for $\sigma \geqq \sigma_e'$.* We have

$$
\begin{aligned}
&[\varphi_{0n}(\sigma^{(i)})]^{-1}(J_i\,\varphi_n^{(i)})(\sigma)\\
(13)\qquad &= [\varphi_{0n}(\sigma^{(i)})]^{-1}(J_i\,\varphi_n^{(i)})(\sigma^{(1)}, \ldots, \sigma_e^{(i)'}, \ldots, \sigma^{(m)})]\\
&\quad + [\varphi_{0n}(\sigma^{(i)})]^{-1}[(J_i\,\varphi_n^{(i)})(\sigma) - (J_i\,\varphi_n^{(i)})(\sigma^{(1)}, \ldots, \sigma_e^{(i)'}, \ldots, \sigma^{(m)})].
\end{aligned}
$$

---

* It should be remembered throughout that σ_e' is an abbreviation for $(\sigma_e^{(1)'}, \sigma_e^{(2)'}, \ldots, \sigma_e^{(m)'})$, and that $\sigma \geqq \sigma_e'$ is an abbreviation for the set of relationships $\sigma_e^{(1)} \geqq \sigma_e^{(1)'}, \ldots, \sigma_e^{(m)} \geqq \sigma_e^{(m)'}$.

Analogous to (18) of Paper I we have the relationship

$$(14)\qquad (J_i\,\varphi_n^{(i)})\,(\sigma) = \left(J_i\left[\frac{1}{n}\,(D_i\,\varphi_{0n\,.\,i})\,\varphi\right]\right)(\sigma).$$

Making use of (14), and postulates $M_2^{(i)}$ and $I_J^{(i)}$, we see that the second term on the right hand side of (13) lies between

$$\frac{1}{n}\left(a-\frac{e}{4}\right)\left[1-\frac{\varphi_{0n}\,(\sigma_e^{(i)'})}{\varphi_{0n}\,(\sigma^{(i)})}\right] \text{and } \frac{1}{n}\left(a+\frac{e}{4}\right)\left[1-\frac{\varphi_{0n}\,(\sigma_e^{(i)'})}{\varphi_{0n}\,(\sigma^{(i)})}\right].$$

From (IV) of Paper I it follows that for a proper choice of $\sigma_e'' > \sigma_e'$ the above expression differs from a/n by a quantity that is less in absolute value than $\frac{1}{2}e$ for all $\sigma > \sigma_e''$.

The first term on the right side of (13) is seen from (14), $M_1^{(i)}$, and $I_J^{(i)}$ to be less in absolute value than

$$\frac{a_1}{n}\left|\frac{\varphi_{0n}\,(\sigma_e^{(i)})}{\varphi_{0n}\,(\sigma^{(i)})}\right|.$$

From IV of Paper I it follows that we can choose $\sigma_e''' > \sigma_e'$ so as to make this expression less in absolute value than $\frac{1}{2}e$ for $\sigma > \sigma_e'''$. If now we choose for σ_e the greater of σ_e'' and σ_e''', it follows from (13) that for $\sigma > \sigma_e$,

$$\left|[\varphi_{0n}\,(\sigma^{(i)})]^{-1}\,(J_i\,\varphi_n^{(i)})\,(\sigma) - (a/n)\right| < e,$$

and the first part of our conclusion is established.

Making use of (14) and $M_1^{(i)}$, we have

$$\frac{-a_1}{n} < [\varphi_{0n}\,(\sigma^{(i)})]^{-1}\,(J_i\,\varphi_n^{(i)})\,(\sigma) < \frac{a_1}{n},$$

which establishes the second part of our conclusion.

LEMMA 3. *If* $\lim_\sigma \varphi(\sigma)$ *exists and is equal to* a, *and* $|\varphi(\sigma)| < a_1$ *for every* σ, *then* $\lim_\sigma (S_n \varphi)(\sigma)$ *will exist and be equal to* a *and we shall have* $|(S_n \varphi)(\sigma)| < a_2$ *for every* σ.

By virtue of definition (10) the operation S_n is equivalent to a succession of operations $S_n^{(i)}$ for $i = 1, 2, \ldots, m$. It follows from Lemma 2, for the case $n = 1$, that if a function $\varphi(\sigma)$ remains finite for all σ and approaches a limit as to σ, the same is true for the function resulting from the operation $S_n^{(i)}$ applied to $\varphi(\sigma)$. Hence by a succession of m applications of Lemma 2, we obtain the conclusion of the present lemma.

We now set

$$\varphi_i'(\sigma) = (S_n^{(i)} \varphi)(\sigma) \qquad (i = 1, 2, \ldots, m). \tag{15}$$

We then prove

LEMMA 4. *If for any* i $\lim_\sigma \varphi_i'(\sigma)$ *exists and is equal to* a, *and* $|\varphi_i'(\sigma)| < a_1$ *for every* σ, *then* $\lim_\sigma \varphi(\sigma)$ *will exist and be equal to* a, *and we shall have* $|\varphi(\sigma)| < a_2$ *for every* σ.

By a procedure analogous to that used in the proof of Lemma 3 of Paper I we may transform equation (15) into the form

$$\varphi(\sigma) = (T_n^{(i)} \varphi_i')(\sigma) \quad (n \geqq 2;\ i = 1, \ldots, m), \tag{16}$$

where $T_n^{(i)}$ is defined by equation (11). Our lemma then follows from Lemma 2 for $n \geqq 2$. For $n = 1$ it is an obvious consequence of (15) and (8).

LEMMA 5. *If* $\lim_\sigma (S_n \varphi)(\sigma)$ *exists and is equal to* a *and* $|(S_n \varphi)(\sigma)| < a_1$ *for every* σ, *then* $\lim_\sigma \varphi(\sigma)$ *will exist and be equal to* a *and we shall have* $|\varphi(\sigma)| < a_2$ *for every* σ.

Making use of the definition of S_n given in equation (10), we see that this lemma may be established by successive applications of Lemma 4.

Noting that S_n and M are interchangeable operations, we have from successive applications of (12), in a manner analogous to the corresponding reductions in Paper I by means of equation (14) of that paper,

$$(H_n \eta)(\sigma) = \Big(S_1\Big(S_2 \cdots \Big(S_3 (S_{n-1} (S_n (C_n \eta))) \cdots \Big)\Big)\Big)(\sigma) \qquad (n). \tag{17}$$

We are now ready to prove our theorem:

THEOREM. *If* $\lim_\sigma (C_n \eta)(\sigma)$ *exists and is equal to* a, *then* $\lim_\sigma (H_n \eta)(\sigma)$ *will exist and be equal to* a, *and conversely.*

From (17), (B), and successive applications of Lemma 3, we obtain the result:

If there exists $\lim_\sigma (C_n \eta)(\sigma) = a$, *then there exists* $\lim_\sigma (H_n \eta)(\sigma) = a\,(n)$.

From (17), (B), and successive applications of Lemma 5, we obtain the result:

If there exists $\lim_\sigma (H_n \eta)(\sigma) = a$, *then there exists* $\lim_\sigma (C_n \eta)(\sigma) = a\,(n)$.

Our theorem is therefore established.

UNIVERSITY OF CINCINNATI,
CINCINNATI, OHIO.

THE EQUILONG TRANSFORMATIONS OF EUCLIDEAN SPACE*

BY

B. H. BROWN

1.

In the classical non-euclidean geometries of space of n dimensions, distance as well as angle has a projective definition, and equilong transformations are the dual of conformal transformations by polar reciprocation in the absolute. In euclidean space the projective definition is lost, but while the preceding duality breaks down, Scheffers† exhibited a perfect analogy in the euclidean plane by the use of the dual numbers of Study. We know that for any function of the complex variable

$$f(x+iy) = X(x, y) + iY(x, y), \qquad i^2 = -1,$$

where X and Y satisfy the Cauchy-Riemann differential equations

$$\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y}, \quad \frac{\partial X}{\partial y} = -\frac{\partial Y}{\partial x}, \tag{1}$$

the point transformation

$$X = X(x, y), \quad Y = Y(x, y) \tag{2}$$

is directly conformal. Scheffers proved that if u and v denote the Hessian normal coördinates of an oriented line (v the distance parameter), for any function of the dual number

$$f(u+\epsilon v) = U(u, v) + \epsilon V(u, v), \qquad \epsilon^2 = 0,$$

where U and V satisfy the differential equations

$$\frac{\partial U}{\partial u} = \frac{\partial V}{\partial v}, \quad \frac{\partial U}{\partial v} = 0, \tag{3}$$

* Presented to the Society, December 27, 1922.

† Mathematische Annalen, vol. 60 (1905), p. 491.

the oriented line transformation

$$U = U(u, v), \quad V = V(u, v)$$

is directly equilong.

Since the conformal group in non-euclidean as well as in euclidean three-space is a ten-parameter group, the equilong group in non-euclidean three-space depends on ten parameters. *But the equilong group in euclidean space contains arbitrary functions.** In space of more than three dimensions, the conformal euclidean group, and the conformal and equilong non-euclidean groups contain a finite number of parameters, but Coolidge† has shown that: *The most general equilong transformation of a euclidean space of n dimensions depends on the most general conformal transformation of a space of $n-1$ dimensions and an arbitrary function of the direction parameters. The distance parameter enters linearly.*

The above theorem is true for $n \geqq 3$, but the last statement is also true for $n = 2$, since the integration of (3) gives

$$U = U(u), \qquad V = U' \cdot v + U_1(u). \tag{4}$$

This fact leads to a hitherto unnoticed analogy between the conformal and equilong transformations in the plane, and to a sharpening of the contrast in higher spaces. The functions X and Y of (1) satisfy Laplace's equation. Again in Study's formulation of the conformal (and therefore equilong) transformations in the Riemannian and Lobatschewskian planes, the functions of hypercomplex variables are separable into functions satisfying either Laplace's equation or the hyperbolic form

$$\frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial y^2} = 0,$$

* This remarkable theorem was first enunciated, without proof, by Study, Sitzungsberichte der Niederrheinischen Gesellschaft für Natur- und Heilkunde, Dec. 5, 1904. In 1908 Coolidge gave the first published proof of this theorem, and a correct explicit form for these transformations in these Transactions, vol. 9 (1908), p. 178. An incorrect derivation leading to a ten-parameter group was given by Loehrl in his Würzburg dissertation (1910). A demonstration, independent of Coolidge's, was given by Blaschke, Archiv der Mathematik und Physik, vol. 16 (1910), p. 182. The final form of these transformations is, however, incorrect with respect to a distinction of signs. This error has never, to our knowledge, been corrected. In 1916 Coolidge in his *Treatise on the Circle and the Sphere,* p. 419, changing the correct form of his 1908 paper, reproduced Blaschke's incorrect form.

† Loc. cit., p. 182.

equations which are not essentially distinct for complex solutions. Finally the functions U and V of (4) satisfy the *parabolic* equation

$$\frac{\partial^2 \Phi}{\partial v^2} = 0. \tag{5}$$

But while the functions in the equations of an equilong transformation in n-dimensional euclidean space are non-trivial solutions of (5) (v denoting the distance parameter), the analogy is completely lost in the other cases.

In this paper we give in Section 2 a new demonstration of the fundamental equations for equilong transformations in euclidean three-space. The main portion of this paper is then devoted to a discussion of groups of these transformations which leave invariant various differential expressions and equations.

2.

The equation

$$\frac{u+v}{1+uv}x - \frac{i(u-v)}{1+uv}y + \frac{1-uv}{1+uv}z = \frac{w}{1+uv} \tag{6}$$

represents an oriented plane such that the direction cosines of its oriented normal are the coefficients of x, y, and z, and such that the distance from the origin to the plane is $\frac{w}{1+uv}$. Then u, v, w are Bonnet* tangential coördinates of this oriented plane. Exceptional cases occur when: (a) $1+uv=0$ (minimal plane); (b) the spherical representation of the plane is a point on a ruling through the south pole. The point of contact of a plane and any envelope which touches the plane is given by the equations

$$\begin{aligned} (u+v)x - i(u-v)y + (1-uv)z &= w, \\ x - iy - vz &= \frac{\partial w}{\partial u} = p, \\ x + iy - uz &= \frac{\partial w}{\partial v} = q. \end{aligned} \tag{7}$$

The square of the distance between two points of a plane is

$$(p_1 - p_2)\,(q_1 - q_2). \tag{8}$$

*Liouville's Journal, ser. 2, vol. 5 (1860), pp. 153–266.

In any plane transformation

$$
(9)\qquad \begin{aligned} U &= U(u, v, w), \\ V &= V(u, v, w), \\ W &= W(u, v, w), \qquad J \neq 0, \end{aligned}
$$

corresponding points of two corresponding planes are projectively related. To find the equations of the equilong transformations we simplify the form of (9) by imposing the necessary conditions that the collineation be

(*a*) affine;

(*b*) directly or indirectly conformal.

Blaschke has shown that under these impositions the once-extended transformations are

$$
\begin{aligned}
&\textit{Direct}: \; U = U(u), && \textit{Indirect}: \; U = U(v), \\
&V = V(v), && V = V(u), \\
&W = W(u, v, w), && W = W(u, v, w), \\
&P = \frac{1}{U'}\left(\frac{\partial W}{\partial w} p + \frac{\partial W}{\partial u}\right), && P = \frac{1}{V'}\left(\frac{\partial W}{\partial w} q + \frac{\partial W}{\partial v}\right), \\
&Q = \frac{1}{V'}\left(\frac{\partial W}{\partial w} q + \frac{\partial W}{\partial v}\right); && Q = \frac{1}{U'}\left(\frac{\partial W}{\partial w} p + \frac{\partial W}{\partial u}\right).
\end{aligned}
$$

It is now necessary and sufficient to impose the condition that

$$
(p_1 - p_2)(q_1 - q_2)
$$

be an absolute invariant. In either case we have

$$
(P_1 - P_2)(Q_1 - Q_2) = \frac{1}{U'V'}\left(\frac{\partial W}{\partial w}\right)^2 (p_1 - p_2)(q_1 - q_2) = (p_1 - p_2)(q_1 - q_2).
$$

Hence

$$
\frac{\partial W}{\partial w} = \sqrt{U'V'}.
$$

We thus have as our fundamental equations

$$
\begin{array}{llll}
 & \textit{Direct}:\ U = U(u), & & \textit{Indirect}:\ U = U(v), \\
(10) & \qquad V = V(v), & & \qquad V = V(u), \\
 & W = \sqrt{U'V'}\, w + F(u, v); & & W = \sqrt{U'V'}\, w + F(u, v).
\end{array}
$$

Blaschke, and subsequent writers, incorrectly insert a $\pm$ sign under the radicals of (10). For such transformations the plane projectivity is either directly conformal and indirectly equiareal, or indirectly conformal and directly equiareal. In neither of these cases is square of distance preserved.

3.

We state, without proof, the fundamental formulas in the differential geometry of a non-developable oriented surface $w = w(u, v)$. The coördinates of a point of tangency are given by

$$
x + iy = \frac{uw - pu^2 + q}{1 + uv},
$$

$$
(11) \qquad x - iy = \frac{vw - qv^2 + p}{1 + uv},
$$

$$
z = \frac{w - up - vq}{1 + uv}.
$$

Let $\dfrac{\partial^2 w}{\partial u^2} = r$, $\dfrac{\partial^2 w}{\partial u\, \partial v} = s$, $\dfrac{\partial^2 w}{\partial v^2} = t$; then the three fundamental forms are

$$
ds^2 = r(z+s)du^2 + \{(z+s)^2 + rt\}\,du\,dv + t(z+s)dv^2,
$$

$$
(12) \qquad -\frac{r}{1+uv}du^2 - \frac{2(z+s)}{1+uv}du\,dv - \frac{t}{1+uv}dv^2,
$$

$$
d\sigma^2 = \frac{4\,du\,dv}{(1+uv)^2}.
$$

The total curvature is

$$
(13) \qquad \frac{-4}{(1+uv)^2\{rt - (z+s)^2\}};
$$

the mean curvature

$$\frac{4(z+s)}{(1+uv)\{rt-(z+s)^2\}}; \tag{14}$$

the lines of curvature are given by

$$r\,du^2 - t\,dv^2 = 0; \tag{15}$$

the radii of principal curvature by

$$R = \frac{-w+up+vq-(1+uv)(s+\sqrt{rt})}{2}; \tag{16}$$

and the centers of curvature by

$$\begin{aligned} X+iY &= q-u(s+\sqrt{rt}), \\ X-iY &= p-v(s+\sqrt{rt}), \\ Z &= \frac{w-up-vq-(1-uv)(s+\sqrt{rt})}{2}. \end{aligned} \tag{17}$$

The differential equations of minimal curves and of asymptotic curves follow from (12). The differential equation of minimal surfaces is

$$z+s=0, \qquad rt \neq 0; \tag{18}$$

of spheres (oriented, non-null spheres)

$$r=t=0, \qquad z+s \neq 0; \tag{19}$$

of points, which are not to be excluded on the score of the discriminant of the first quadratic form vanishing, but which are proper envelopes of ∞^2 planes, and may be regarded either as minimal surfaces or as spheres,

$$r=t=z+s=0. \tag{20}$$

4.

Among the indirect equilong transformations, the analogue of the identity is that one which merely reverses the orientation of a plane, without changing

its position. This transformation we term the "pseudo-identity". It is clear that every indirect transformation is the product of a direct transformation and the pseudo-identity. The twice-extended form of the pseudo-identity is

$$
\begin{aligned}
U &= -\frac{1}{v}, \\
V &= -\frac{1}{u}, \\
W &= -\frac{w}{uv}, \\
P &= \frac{w - vq}{u}, \\
Q &= \frac{w - up}{v}, \\
R &= -\frac{v^3 t}{u}, \\
S &= up + vq - w - uvs, \\
T &= -\frac{u^3 r}{v}.
\end{aligned}
\tag{21}
$$

It will be observed that the equations and expressions (11) to (20) are invariant (invariant except for sign) under (21) as they have geometric significance independent of (dependent on) orientation. *The differential geometry of oriented surfaces is the interpretation of the differential invariants of the extended pseudo-identity.* A surface whose equation is invariant under the pseudo-identity is obviously *one-sided.* We have then the

THEOREM. *A necessary and sufficient condition that a surface $w = f(u, v)$ be one-sided is that f satisfy the functional equation* $f(u, v) = -uv\, f\left(-\frac{1}{v}, -\frac{1}{u}\right)$.

5.

We shall next prove the

THEOREM. *Any oriented non-developable surface may, by each of two and only two distinct, direct equilong transformations, be transformed into any other oriented non-developable surface, and that with an arbitrary analytic directly conformal mapping of their spherical representations.*

This theorem was suggested by Study in his 1904 paper, but he stated, incorrectly, that there was one and only one such transformation.

Let us consider two surfaces

$$W = f_1(U, V),$$

$$w = f_2(u, v),$$

and let us assume that the spherical representation of the first (U, V) is conformally mapped on the spherical representation of the second (u, v) by the directly conformal transformation

$$U = U(u),$$

$$V = V(v).$$

The theorem is proved if, in the group of transformations

$$U = U(u),$$

$$V = V(v),$$

$$W = \sqrt{U'V'}\,w + F(u, v),$$

we can determine two and only two functions $F(u, v)$ such that the first surface is transformed into the second. This means that

$$\sqrt{U'V'}\,w + F(u, v) = f_1(U(u), V(v))$$

must be identical with

$$w = f_2(u, v),$$

which is true when and only when

$$F(u, v) = f_1(U(u), V(v)) - \sqrt{U'V'}\,f_2(u, v);$$

hence there are always two distinct transformations. We should note that there is no exception when $f_2 = 0$.

6.

[1] The twice-extended form of the general direct equilong transformation may be written

$$U = U(u),$$

$$V = V(v),$$

$$W = \sqrt{U'V'}\,w + F(U, V).$$

(22)

$$P = \frac{wV'^{\frac{1}{2}}U''}{2U'^{\frac{3}{2}}} + \frac{pV'^{\frac{1}{2}}}{U'^{\frac{1}{2}}} + F_U,$$

$$Q = \frac{wU'^{\frac{1}{2}}V''}{2V'^{\frac{3}{2}}} + \frac{qU'^{\frac{1}{2}}}{V'^{\frac{1}{2}}} + F_V,$$

$$R = \frac{wV'^{\frac{1}{2}}U'''}{2U'^{\frac{5}{2}}} - \frac{3}{4}\frac{wU''^2V'^{\frac{1}{2}}}{U'^{\frac{7}{2}}} + \frac{rV'^{\frac{1}{2}}}{U'^{\frac{3}{2}}} + F_{UU},$$

$$S = \frac{wU''V''}{4(U'V')^{\frac{3}{2}}} + \frac{pV''}{2U'^{\frac{1}{2}}V'^{\frac{3}{2}}} + \frac{qU''}{2V'^{\frac{1}{2}}U'^{\frac{3}{2}}} + \frac{s}{(U'V')^{\frac{1}{2}}} + F_{UV},$$

$$T = \frac{wU'^{\frac{1}{2}}V'''}{2V'^{\frac{5}{2}}} - \frac{3}{4}\frac{wV''^2U'^{\frac{1}{2}}}{V'^{\frac{7}{2}}} + \frac{tU'^{\frac{1}{2}}}{V'^{\frac{3}{2}}} + F_{VV}.$$

It is proposed to discuss the invariance of the equations and expressions (11) to (20) under (22).

First, under a direct transformation, a necessary and sufficient condition that spheres transform into spheres is that R vanish with r and T with t. This requires

(23)

$$2U'U''' - 3U''^2 = 0,$$

$$2V'V''' - 3V''^2 = 0,$$

$$F_{UU} = F_{VV} = 0.$$

The first two of (23) recall the Schwarzian derivative. Integrating, we have

(24)

$$U = \frac{\alpha u + \beta}{\gamma u + \delta}, \qquad V = \frac{\alpha' v + \beta'}{\gamma' v + \delta'},$$

$$F = AUV + BU + CV + D.$$

We may and shall choose ratios so that

$$\alpha\delta-\beta\gamma=\alpha'\delta'-\beta'\gamma'=1.$$

We have then

$$W=\frac{\pm w+auv+bu+cv+d}{(\gamma u+\delta)(\gamma' v+\delta')}, \tag{25}$$

(24) and (25) giving the equations of the well known Laguerre group. We might just as easily have found these by imposing the condition that lines (more properly strips) of curvature go into lines of curvature.

It is easy to verify that translations are given by

$$U=u,\quad V=v,\quad W=w+a(uv-1)+bu+cv; \tag{26}$$

reflection in the origin by

$$U=u,\quad V=v.\quad W=-w; \tag{27}$$

dilatations by

$$U=u.\quad V=v,\quad W=w+a(uv+1); \tag{28}$$

rotations by

$$U=\frac{au+b}{cu+d},\quad V=\frac{dv-c}{-bv+a},\quad W=\frac{w}{(cu+d)(-bv+a)}. \tag{29}$$

where $ad-bc=1$. Other transformations involve Laguerre inversions.

On account of the simplicity and frequency of occurrence of the expression $z+s$, we next consider the transformations under which minimal surfaces transform into minimal surfaces. It is clear that any transformation

$$U=u.\quad V=v,\quad W=w+f_1(u,v),$$

where f_1 is a solution of $z+s=0$, will carry any minimal surface into a minimal surface, for the sum of two solutions of a linear homogeneous partial differential equation is itself a solution. To this group of transformations we may, from geometric considerations, adjoin (27) and (29). It turns out that these are the only such transformations; we term this group the "minimal group".

To prove this statement, if we impose $Z+S=0$ on $z+s=0$ we must have

$$
\begin{aligned}
(30) \qquad & w\Big[U'^2 V'^2 - \frac{1}{2} U V'^2 U'' - \frac{1}{2} V U'^2 V'' + \frac{1}{4} U'' V'' + \frac{1}{4} U'' V'' U' V'\Big] \\
& + p\Big[- U U' V'^2 + \frac{1}{2} U' V'' + \frac{1}{2} U U' V V''\Big] \\
& + q\Big[- V V' U'^2 + \frac{1}{2} V' U'' + \frac{1}{2} V V' U U''\Big] \\
& + s\,[U' V' + U V U' V'] \\
& + (U' V')^{\frac{3}{2}} [F - U F_U - V F_V + (1 + U V) F_{UV}] \\
& \equiv \frac{U' V' (1 + U V)}{1 + uv} \{w - up - vq + (1 + uv) s\}.
\end{aligned}
$$

Hence it is necessary that

$$
(31) \qquad -\frac{UV'}{1+UV} + \frac{1}{2}\frac{V''}{V'} = -\frac{u}{1+uv},
$$

$$
(32) \qquad -\frac{VU'}{1+UV} + \frac{1}{2}\frac{U''}{U'} = -\frac{v}{1+uv},
$$

$$
(33) \qquad \frac{U'' V''}{4 U' V'} + \frac{U' V'}{1+UV} - \frac{1}{2}\left\{\frac{UV'^2 U'' + VU'^2 V''}{U' V'(1+UV)}\right\} = \frac{1}{1+uv},
$$

$$
(34) \qquad F - UF_U - VF_V + (1+UV) F_{UV} = 0.
$$

Subtracting (33) from the product of (31) and (32) we have

$$
(35) \qquad \frac{U' V'}{(1+UV)^2} = \frac{1}{(1+uv)^2}.
$$

hence we may rewrite (31) and (32) as

$$
(36)\qquad
\begin{aligned}
u &= \frac{UV'^{\frac{1}{2}}}{U'^{\frac{1}{2}}} - \frac{1}{2}(1+UV)\frac{V''}{V'^{\frac{3}{2}}\,U'^{\frac{1}{2}}},\\
v &= \frac{VU'^{\frac{1}{2}}}{V'^{\frac{1}{2}}} - \frac{1}{2}(1+UV)\frac{U''}{U'^{\frac{3}{2}}\,V'^{\frac{1}{2}}}.
\end{aligned}
$$

Differentiating the first of (36) with regard to v, and the second with regard to u,

$$
(37)\qquad
\begin{aligned}
&\frac{U(1+UV)}{V'^{\frac{5}{2}}}\left\{V'''V' - \frac{3}{2}V''^2\right\} = 0,\\
&\frac{V(1+UV)}{U'^{\frac{5}{2}}}\left\{U'''U' - \frac{3}{2}U''^2\right\} = 0;
\end{aligned}
$$

hence

$$
U = \frac{\alpha u+\beta}{\gamma u+\delta}, \qquad V = \frac{\alpha' v+\beta'}{\gamma' v+\delta'}
$$

are *necessary* conditions on U and V. Now if, in (33), we substitute the value of $(1+uv)$ given in (35), and these last values of U and V, it is necessary that

$$
\alpha:\beta:\gamma:\delta = \delta':-\gamma':-\beta':\alpha',
$$

and the theorem is proved.

If we examine the form of any one of (12), (13), or (14) we see without difficulty that any one of them is invariant if and only if the transformation belongs to both the Laguerre and minimal groups. The actual verification of this is so much a repetition of the previous proof that it is omitted. Unfortunately the only such transformations are the congruent transformations; for the only non-parallel transformations of the minimal group are rotations, and of the parallel transformations of the Laguerre group, dilatations carry a point into a sphere which is not a minimal surface. Our results are then essentially negative if we impose the condition on all surfaces, but there are interesting special cases for groups of transformations and groups of surfaces.

We consider only one such example: the conformal mapping of surfaces under equilong transformations. The differential equation of minimal curves on a surface is, by (12),

$$r(z+s)du^2+\{(z+s)^2+rt\}du\,dv+t(z+s)dv^2 = 0.$$

The following theorems follow immediately:

1. *Under a transformation of the Laguerre group any sphere and the transformed sphere are conformally mapped.*

2. *Under a transformation of the minimal group, any minimal surface and its transformed minimal surface are conformally mapped.*

3. *The only surfaces transformed into their spherical representations with conformal mapping by equilong transformations are spheres and minimal surfaces.*

4. *A minimal surface may be transformed into any sphere with conformal mapping, and conversely.*

7.

Since the solutions of linear homogeneous partial differential equations possess the additive property, we may associate with every such equation the surfaces that are solutions thereof, and a corresponding group of direct parallel equilong transformations

$$U = u, \quad V = v, \quad W = \pm w + f(u, v),$$

where f is itself a solution of the given differential equation; under this group of transformations the surfaces are permuted among themselves. Obviously we shall be most interested in differential equations invariant under the pseudo-identity. For such differential equations there is a sub-group of transformations which will permute the one-sided surfaces of the group among themselves, for the solutions of the linear homogeneous functional equation

$$f(u,v) = -uvf\left(-\frac{1}{v}, -\frac{1}{u}\right)$$

possess the additive property. Thus, for example, the double minimal surfaces are permuted among themselves by the appropriate subgroup, for they are the only one-sided minimal surfaces.

Let us consider a solution of (18), a minimal surface whose equation may be written

$$w = 2vf(u)+2uf_1(v)-(1+uv)\,[f'(u)+f_1'(v)].$$

We may associate with this a two-parameter family of minimal surfaces $[A, B]$

$$w = 2vAf + 2uBf_1 - (1+uv)\,[Af' + Bf_1'],$$

where $[A, A]$ are expansions of the original surface, and $[A, 1/A]$ its continuous deforms;* and we may also associate therewith the two-parameter family of parallel equilong transformations of the minimal group which permute these among themselves.

The ∞^2 points, one from each of these surfaces, with properly parallel tangent planes (planes with the same u and v) may be obtained by expanding at the origin the conic which is the path-curve of the point of the original minimal surface under the continuous transformation which gives the associated surfaces. These points are coplanar, as the plane of the conic contains the origin. The tangent planes are not, in general, coincident with this locus plane. Under a parallel equilong transformation any aggregate of ∞^3 planar elements with (properly) parallel planes are rigidly translated as a whole (Study and Blaschke) so that an equilong transformation of the group effects a translation of this plane. In this plane associated minimal surfaces are represented by points of a conic which is one of a one-parameter family of homothetic conics. A transformation of the group will translate this conic, the transformed conic cutting the original conic and each of the ∞^1 homothetic conics in two points (since the axes of the conics are parallel). Hence under any transformation of the group only two associated surfaces of a given minimal surface will transform into associated surfaces; the other associated surfaces will, in general, be transformed by pairs into associated minimal surfaces of the ∞^1 expansions of the given minimal surface.

This discussion may obviously be extended to any system of surfaces whose equations are

$$w = A\sum f_i(u,v)F^{[i]}(u) + B\sum g_i(u,v)G^{[i]}(v), \tag{38}$$

except that the surfaces $[A, 1/A]$ are not generally continuous deforms.

As an example we have certain surfaces of Goursat,† for which the sum of the radii of curvature at a point is proportional to the distance from the

* Although not coextensive, we shall use the expression "continuous deforms" as equivalent to "associated surfaces". This is proper, since a continuous deform is an associated surface, or can be made to coincide with one by a congruent transformation.

† American Journal of Mathematics, vol. 10 (1887-8), p. 187; Baroni, Giornale di Matematiche, vol. 28 (1890), p. 349.

origin to the tangent plane at the point. From (6) and (16) the equation of these surfaces is

$$(39)\qquad (1+uv)s-up-vq+w\left[1+\frac{2k}{1+uv}\right]=0,$$

the ratio of proportionality being $2k$. Goursat proved that when (and only when) m defined by

$$k=\frac{(m+1)(m-2)}{2}$$

is integral, the solution of (39) may be obtained free from quadratures, and, indeed, in the form (38). For these values of k the preceding discussion holds, with deletion of the expression "continuous deforms".

Two special cases are worthy of note:

(*a*) If $k=0$, (39) is the differential equation of minimal surfaces;

(*b*) If $k=-1$, (39) is the differential equation of Appell* surfaces for which the projection of the origin on every normal is midway between the centers of principal curvature.

In the papers of Appell and Goursat, we find three classical transformations:

(*a*) A transformation of Appell which carries a particular minimal surface into an Appell surface.

(*b*) A transformation of Appell which carries a particular Bonnet† surface into an Appell surface;

(*c*) A transformation of Goursat which carries a particular Goursat surface into another Goursat surface with change of k. These transformations are equilong, and, in fact, special cases of Study's theorem where the mapping of the spherical representations is the identity, and the upper sign for the radical is used.

The general methods of this section are applicable to a large class of surfaces defined by some relation involving their radii of curvature. One further group of transformations, defined by a linear non-homogeneous partial differential equation, merits attention. From (16) it follows immediately that

$$(40)\qquad (1+uv)s-up-vq+w=-2k$$

* American Journal of Mathematics, vol. 10 (1887-8), p. 175.

† Paris Comptes Rendus, vol. 42 (1856), p. 119, *Note sur les surfaces pour lesquelles la somme des deux rayons de courbure principaux est égale au double de la normale.*

is the differential equation of all surfaces for which the sum of the radii of principal curvature is a constant $2k$. Such a surface is, for example, the "inner" surface of a sphere of radius k, center at the origin

$$w = -k(1 + uv). \tag{41}$$

Thus knowing one particular solution of (40) we obtain all the other solutions by adding to the right-hand side of (41) the general solution of the differential equation for minimal surfaces. Hence the direct parallel equilong transformations

$$U = u, \quad V = v, \quad W = w + f(u, v),$$

where $f(u, v)$ satisfies (40), carry minimal surfaces into the surfaces we are considering, and carry surfaces the sum of whose radii of principal curvature is $2k_1$ into surfaces the sum of whose radii of principal curvature is $2(k + k_1)$.

DARTMOUTH COLLEGE,
HANOVER, N. H.

INVARIANT SETS OF EQUATIONS IN RIEMANN SPACE*

BY

PHILIP FRANKLIN

1. Introduction

It is assumed in the theory of relativity that physical quantities are represented by expressions derived from tensor components, and that the laws of nature may be expressed as equations stating the equality of two tensors. This assumption is made so as to satisfy the requirement that physical laws must be expressible in a form independent of the particular coördinates used. If we start with this latter assumption, we are tempted to require merely that the equations expressing a law of nature be invariant, as a set, under transformations of coördinates, and the question arises as to the relation of equations of this type, referred to in the sequel as an *invariant set* of equations, to the tensor equations usually assumed. The requirement of invariance implies that there is a law of transformation for the equations in terms of the transformation of coördinates, which will be given if the equations involve merely tensor components and the coördinates. If these quantities enter into the equations in a sufficiently simple manner (which, however, is as general as is required in most of the equations of physics), we may completely answer the question raised above by the theorem

An invariant set of equations whose members are formed from the components of one or more tensors and point functions by addition, multiplication, and differentiation with respect to the coördinates is equivalent to a set of tensor equations.

Here, as throughout this paper, the tensors relate to a Riemann n-space.

We apply this theorem to the classification of invariant equations involving the derivatives of the fundamental quadratic tensor (g_{ij}) and show that

An invariant set of equations involving the derivatives of the g_{ij}, the second derivatives appearing linearly, and no derivative higher than the second occurring, is equivalent to one of five standard tensor equations.

A theorem nearly equivalent to this for 4-space was given by G. D. Birkhoff.† Besides holding for n-space, our discussion is free from the assumptions that the equations are homogeneous, which rules out Einstein's "cosmological equation", and that the coördinates have a certain reality character.

* Presented to the Society, April 28, 1923.

† *Relativity and Modern Physics*, Harvard University Press, 1923, pp. 211–220.

The above theorems constitute the chief results of this paper. We proceed to the proofs.

2. Equations linear in a single tensor

We shall begin with the special case of equations involving the components of a single tensor, and these linearly, and shall prove that such an invariant set is equivalent to a set of tensor equations, the left members of these equations being linear combinations of the single tensor given, and tensors easily derived from it, with scalar coefficients. To fix the ideas, we shall give the details in full only for a tensor of the fourth order; the methods are, however, general.

If we consider our equations at a single point, the coefficients of the tensor components, which in general are point functions, become constants. If we further introduce normal (orthogonal geodesic) coördinates at this point, we have there

$$g_{ij} = \delta_{ij}, \qquad \partial g_{ij}/\partial x_k = 0, \tag{1}$$

and the transformations of coördinates which change one such system into another are those belonging to the orthogonal group. Since our equations remain invariant under all coördinate transformations, they retain their form (as a set) for any linear orthogonal transformation. This leads to the

LEMMA. *An invariant set of equations linear in a single tensor is equivalent to a set of equations each of which has the property that any subscript appears an odd number of times in every one of its terms, or an even number, perhaps zero.*

We prove this by noting that when we perform the transformation

$$x_1' = -x_1, \qquad x_i' = x_i \qquad (i \neq 1), \tag{2}$$

which corresponds to a reflection in the 1-axis, any term containing the subscript 1 an odd number of times has its sign changed, while any one containing it an even number of times is unaffected. Hence if we apply the reflection (2) to any one of our equations, we obtain a new equation which is a consequence of our given set, from the invariant character, and on being added to and subtracted from the original equation gives rise to two equations of the type required by the lemma for the subscript 1. By repeating the process for the remaining subscripts, we reach the desired result.

We turn now to the set of equations linear in the components of P_{abcd}, the tensor of the fourth order. Suppose first that some one of our equations contains a term with four distinct subscripts, say 1234 (all the subscripts, if in four-space, a particular group if in n-space ($n > 4$); if in a space of less than four dimensions, there are no such terms, and our argument proceeds at once

to the place below where terms with subscripts not all distinct are discussed). By applying the lemma, we may then obtain an equation in which every term contains these four distinct subscripts an odd number of times, and therefore each once. Let this equation be

$$AP_{1234} + BP_{1342} + \cdots = 0. \tag{3}$$

If we define the new tensor

$$Q_{abcd} = AP_{abcd} + BP_{acdb} + \cdots, \tag{4}$$

we see that (3) gives

$$Q_{1234} = 0. \tag{5}$$

Since our set of equations is invariant, (3) and hence (5) holds in all systems of coördinates. In particular, since we may transform the coördinates by a permutation of the variables, we see that all the components of Q_{abcd} with four distinct subscripts vanish.

We shall obtain the tensor equation which follows from our set by determining constants A, B, etc. for which the equation

$$\begin{aligned} Q_{abcd} = {} & \delta_{ab}\,(A Q_{iicd} + B Q_{iidc} + C Q_{icdi} + D Q_{idci} + E Q_{icid} + F Q_{idic} \\ & + G Q_{cdii} + H Q_{dcii} + I Q_{ciid} + J Q_{diic} + K Q_{cidi} + L Q_{dici}) \\ & + \delta_{ac}\,(A' Q_{iibd} + B' Q_{iidb} + \cdots) + \cdots \\ & + \delta_{ab}\,\delta_{cd}\,(U Q_{iijj} + V Q_{ijji} + W Q_{ijij}) + \cdots \\ & + \delta_{ac}\,\delta_{bd}\,(U' Q_{iijj} + V' Q_{ijji} + W' Q_{ijij}) + \cdots \end{aligned} \tag{6}$$

is satisfied.

This equation holds regardless of the values of the constants, provided all four subscripts are distinct, in virtue of (5) and the similar equations. To see what conditions must be satisfied when two subscripts become equal, note that the equations

$$\begin{aligned} x_1' &= x_1 \cos\theta - x_2 \sin\theta, \\ x_2' &= x_1 \sin\theta + x_2 \cos\theta, \qquad x_i' = x_i \qquad (i \neq 1, 2) \end{aligned} \tag{7}$$

define an admissible transformation, and therefore our set of equations will hold after this transformation is applied. On applying it to (5), we obtain an equation in tan θ true for all values of θ. Consequently the coefficients must vanish, giving the new equation

$$Q_{1134} = Q_{2234}. \tag{8}$$

On applying the rotation similar to (7), but involving the 1 and 3-axes, to equation (8), which likewise holds for all systems of coördinates, multiplying the right side by $\cos^2\theta + \sin^2\theta$ to make the equation homogeneous, equating the coefficients to zero, and using (5), (8) and the similar equations, we find

$$Q_{1114} = Q_{1224} + Q_{2124} + Q_{2214}. \tag{9}$$

If we now set $abcd = 1123$ in (6), and make use of (8), (9) and the similar equations, we obtain

$$\begin{aligned} Q_{1123} = {} & Q_{1123}(nA + D + E + I + L) + Q_{2311}(D + F + nG + I + K) \\ & + Q_{1132}(nB + C + F + J + K) + Q_{3211}(C + E + nH + J + L) \\ & + Q_{1231}(B + nC + E + H + K) + Q_{2113}(A + E + G + nI + K) \\ & + Q_{1321}(A + nD + F + G + L) + Q_{3112}(B + F + H + nJ + L) \\ & + Q_{1213}(A + C + nE + H + I) + Q_{2131}(B + C + G + I + nK) \\ & + Q_{1312}(B + D + nF + G + J) + Q_{3121}(A + D + H + J + nL). \end{aligned} \tag{10}$$

The equations obtained by making this an identity in the Q's and equating coefficients have a determinant equal to $n^3(n-2)^5(n+4)(n+2)^3$ which is different from zero (since $n \geqq 4$, to make this part of the argument necessary). Hence the equations have a solution, and when they are solved and the result is substituted in (6) it will hold for $abcd = 1123$. It will also hold for any choice of the subscripts making the first two equal, and the remaining two distinct from these and from each other, as is evident from the equations determining the constants.

In an entirely analogous way, we may determine the values of $A', B', \ldots, L'$ so that the equation (6) will hold when a and c are equal, b and d being distinct from these and from each other; and then $A'', \ldots, A''', \ldots, A^{vi}, \ldots$ so that the equation will hold when any pair of subscripts are equal, the remaining

pair being distinct from these and each other. It then follows from (9) and the similar equations that (6) holds when three of the subscripts are equal, but different from the fourth.

We now define a tensor obtained from Q_{abcd} by subtracting the terms already determined:

$$S_{abcd} = Q_{abcd} - \delta_{ab}\, A\, Q_{iicd} - \cdots - \delta_{cd}\, L^{\mathrm{vi}}\, Q_{biai}. \tag{11}$$

From the method we used to determine the constants appearing in this equation, it follows that all the components of S_{abcd} containing one of the subscripts an odd number of times vanish. Furthermore, by contracting (11) we may express the contracted S's in terms of the contracted Q's. Consequently we may find U, V, W etc. to satisfy (6) provided we can find constants which satisfy

$$S_{abcd} = \delta_{ab}\, \delta_{cd}\, (X S_{iijj} + Y S_{ijji} + Z S_{ijij}) + \cdots. \tag{12}$$

We already know that the equations

$$S_{1234} = 0, \tag{13}$$

$$S_{1123} = 0, \tag{14}$$

$$S_{1114} = 0 \tag{15}$$

hold, as well as the similar equations obtained from them by permuting the subscripts. On applying the rotation similar to (7) involving the 2- and 3-axes to (14), and noting that we have an identity in θ, we find

$$S_{1122} = S_{1133}, \tag{16}$$

and hence

$$S_{1122} = S_{3344}. \tag{17}$$

Again, by applying the rotation involving the 1- and 4-axes, and using the equations already set down, we find

$$S_{1111} = S_{4114} + S_{1144} + S_{1414}, \tag{18}$$

and hence, by (16),

$$S_{1111} = S_{2222}. \tag{19}$$

If we now set $abcd = 1122$ in (12) and utilize the relations just derived, we find that

$$
\begin{aligned}
S_{1122} &= S_{1122}(n^2X + nY + nZ) + S_{1221}(nX + n^2Y + nZ) \\
&\quad + S_{1212}(nX + nY + n^2Z).
\end{aligned} \tag{20}
$$

The equations obtained by considering this equation an identity in S have a determinant equal to $n^3(n+2)(n-1)^2$, different from zero ($n \geqq 2$, since we have two distinct subscripts). These equations may thus be solved for X, Y and Z. Similarly we determine X', Y', Z' so that (12) holds for S_{1221}, and X'', Y'', Z'' so that it holds for S_{1212}. It then follows from the way these coefficients are determined that equation (12) holds for any component whose subscripts form two pairs of equal elements, all four not being the same. This last case, however, is covered by recalling (18) and noticing the form of (12).

Having thus determined the coefficients to satisfy (12), which is now true for all subscripts, we combine (12), (11) and the equations obtained by contracting (11) so as to get an equation of the form (6) which is true for all subscripts. On eliminating Q_{abcd} from this equation by means of (4) we obtain a tensor equation in P_{abcd} which is implied by our original set. If there are any other equations in the set in four distinct subscripts, which are not consequences of the tensor equation just obtained, we may apply the process again to get an additional tensor equation in P_{abcd}. We may keep this up until the tensor equations obtained have as consequences all the equations of our set with four distinct subscripts. This will have to happen after at most 4! such tensor equations have been obtained, since from this number we could solve for all the components P_{1234}, P_{1324} etc., and eliminate them from our original set.

When we have obtained these tensor equations, and removed from our original set of equations all those which are consequences of the tensor equations, we shall have a second set of equations. This residual set has the property that no member contains four distinct subscripts. If one of these equations contains one, and hence two indices (say 2,3) each an odd number of times, by applying the lemma we may obtain from it an equation in which every term contains these indices an odd number of times, and hence of the form

$$
m_{1123}P_{1123} + \cdots + m_{2223}P_{2223} + \cdots = 0, \tag{21}
$$

where the m's are mere numbers, coefficients of the components indicated by their subscripts.

On applying the rotation similar to (7) involving the 1- and 4-axes, to (21) and equating the coefficient of $\sin\theta$ to 0, we find

$$(22)\qquad (m_{4423}-m_{1123})\,P_{1423}+\text{terms in less than 4 subscripts}=0.$$

If the coefficient of P_{1423} is not zero, the equation

$$(23)\qquad P_{1423}=0,$$

and hence (cf. (8))

$$(24)\qquad P_{4423}=P_{1123}$$

will be included in the consequences of our tensor equations derived from those involving four subscripts, and hence in view of (24) we can make

$$(25)\qquad m_{4423}=m_{1123}$$

in all cases.

If we next apply (7), involving the 1- and 2-axes, to (21), we find, as the condition for the coefficient of $\sin\theta\cos^2\theta$ vanishing,

$$(26)\qquad (m_{2223}-m_{1123}-m_{1213})\,P_{1223}+m_{2443}\,P_{1443}+\cdots=0.$$

As this is of type (21), 1 and 3 being the odd subscripts, we obtain the relation, analogous to (25),

$$(27)\qquad m_{2223}-m_{1123}-m_{1213}=m_{2443},$$

or, in view of the relations similar to (25),

$$(28)\qquad m_{2223}=m_{1123}+m_{2113}+m_{1213}.$$

In consequence of equations (25), (28) and similar equations, we may write (21) in the form

$$(29)\qquad m_{1123}\,P_{ii23}+m_{1231}\,P_{i23i}+\cdots=0,$$

which shows that the tensor

$$(30)\qquad T_{ab}=m_{1123}\,P_{iiab}+m_{1231}\,P_{iabi}+\cdots$$

has all its components in two distinct subscripts zero, and by the method used above for Q_{abcd}, we may show that T_{ab} satisfies the tensor equation

$$T_{ab} = \frac{1}{n}\delta_{ab}\, T_{ii}. \tag{31}$$

On eliminating T_{ab} from (31) by means of (30) we obtain a tensor equation in P_{abcd}. In the same way we obtain all possible tensor equations which result from the equations of our set with two distinct subscripts. We then reject all equations from the set which are consequences of any of the tensor equations so far obtained. The equations which remain, since they involve no subscript an odd number of times, must be of the form

$$m_{1122}\, P_{1122} + \cdots + m_{1111}\, P_{1111} + \cdots = 0. \tag{32}$$

On applying the rotation similar to (7) involving the 1- and 3-axes, we find, as the coefficient of $\sin\theta \cos^3\theta$,

$$(m_{1122} - m_{3322})P_{1322} + (m_{1111} - m_{3113} - m_{1313} - m_{1133})P_{1113} + \cdots = 0. \tag{33}$$

By an argument similar to that used above to establish (25) we may show that the coefficients here either are zero, or can be made zero, giving

$$m_{1122} = m_{3322}, \tag{34}$$

$$m_{1111} = m_{3113} + m_{1313} + m_{1133}. \tag{35}$$

Consequently we may write (32) in the form

$$m_{1122}\, P_{iijj} + m_{1221}\, P_{ijji} + \cdots + M = 0, \tag{36}$$

M being a constant term. This is a tensor equation, and by obtaining all such equations from our set we will finally have a collection of tensor equations which is equivalent to our set, in the sense that it implies and is implied by the set. These equations are either of type (6), (31) using (30), or (36). All these may be included in a single form,

$$A\, P_{abcd} \cdots + B\delta_{ab}\, P_{iicd} \cdots + C\delta_{ab}\,\delta_{cd}\, P_{iijj} \cdots + \delta_{ab}\,\delta_{cd}\, D = 0. \tag{37}$$

The coefficients in this equation are constants at the point under consideration, and the equation holds for normal coördinates at this point. As it is evidently equivalent to the equation in general coördinates

$$(38) \quad A P_{abcd} \cdots + B g_{ab} P_{iicd} \cdots + C g_{ab} g_{cd} P_{iijj} \cdots + g_{ab} g_{cd} D = 0,$$

this may now be considered to hold at all points, if instead of regarding the coefficients as constants, we regard them as scalar point functions.

As the argument given above for tensors of the fourth order may evidently be extended to those of any order, the mth, the process consisting in first deducing tensor equations from those in the set in m distinct subscripts, then $m-2$, and so on, we may state as the conclusion of this section

THEOREM I. *An invariant set of equations, linear in the components of a single tensor, is equivalent to a set of tensor equations, obtained by equating to zero linear combinations of the given tensor, those obtained from it by permuting the subscripts, and those obtained by contracting one or more times and multiplying by the fundamental quadratic tensor so as to bring the order to its original value. The coefficients in the linear relations are scalar point functions.*

3. Extension to the general case

The previous section merely dealt with equations linear in the components of a single tensor; it is, however, easy to extend the result there obtained to the case of equations formed from several tensors, differentiation and multiplication being admitted. This extension now concerns us.

Consider first the case where the equations are linear in the components of several tensors, not necessarily of the same order, so that each term involves only a component of one of the tensors. We confine our attention to one point, and introduce normal coördinates. The lemma of the preceding section evidently applies here, and by its use we may reduce our set to one in which each equation contains any one subscript an odd number of times, or an even number of times. Hence the tensors appearing in these equations must have orders differing by an even number, and they may all be brought up to the same order by introducing δ's with equal numerical subscripts, which does not effect the invariant character of the set. Having done this, we may now select the equation of the set containing the greatest number of distinct subscripts, as we did in the preceding paragraph, and proceed to deduce tensor equations which follow from our set, exactly as before.

Next, if the equations of the set express the vanishing of expressions involving the components of several tensors to a degree higher than the first, being polynomials in these components, we have merely to regard the product

of two components of the same, or of two different tensors, as a single component of a tensor whose order is the sum of the orders of the two used in forming it, to reduce this to the case just discussed.

Finally we consider the case where differentiation of the tensor components is permitted. We notice that we are dealing with normal coördinates in carrying out our reduction, and in these coördinates the first derivatives may be replaced by the corresponding covariant derivatives, while the higher derivatives may be replaced by polynomials in the higher covariant derivatives, and the derivatives of the Christoffel symbols in normal coördinates, evaluated at the origin. But these quantities are all tensors,* and thus the set of equations implies a set holding in normal coördinates which may be treated by methods already given.

As an illustration of the reduction of the last three paragraphs, suppose one of our original equations were

$$(W_{12})^3 = U_{12} + (V_{12})(\partial T_1/\partial x_2). \tag{39}$$

By applying the lemma, we should obtain, after replacing the derivative by a covariant derivative,

$$\begin{aligned}(V_{12})(T_{1/2}) &= 0,\\ (W_{12})^3 - U_{12} &= 0,\end{aligned} \tag{40}$$

and on introducing the tensors defined by

$$\begin{aligned}Q_{abcd} &= V_{ab}\,T_{c/d},\\ Q'_{abcdef} &= W_{ab}\,W_{cd}\,W_{ef} - \delta_{ce}\,\delta_{df}\,U_{ab},\end{aligned} \tag{41}$$

we should obtain

$$\begin{aligned}Q_{1212} &= 0,\\ Q'_{121212} &= 0,\end{aligned} \tag{42}$$

a form to which the result of the preceding section would apply.

We have thus proved the theorem stated in the introduction:

THEOREM II. *An invariant set of equations, obtained by equating to zero expressions formed from one or more given tensors and point functions by*

* For the expression of the derivatives of the Christoffel symbols at the origin of a system of normal coördinates in terms of the curvature tensor, see O. Veblen, Proceedings of the National Academy of Sciences, vol. 8 (1922), p. 196.

addition, multiplication, and differentiation with respect to the coördinates, is equivalent to a set of tensor equations.

4. Equations linear in the second derivatives of the fundamental quadratic tensor

In this section we shall consider the classification of invariant sets of equations, formed by equating to zero expressions involving the fundamental quadratic tensor, g_{ij}, its first and second derivatives, and these last linearly. Such equations are of interest, since the equations holding in space free of matter are of this form, so that on specializing our results to four-space they will throw light on the choice of equations for the relativistic theory of gravitation. The question is related to one concerning possible tensors of this type, which was previously taken up by the author.* The results of that paper, while related to those here obtained, neither follow from them nor lead to them.

As our invariant set of equations involve merely the g_{ij}, and their first and second derivatives, if we introduce normal coördinates, for which (1) holds, they will reduce to expressions in the second derivatives, and as these (in normal coördinates) are expressible in terms of the curvature tensor, we see that our equations involve this tensor only. Also, on account of the linearity requirement, they involve its components linearly, so that they are the type discussed in Section 2. By that section, we see that our equations must all be of the form

$$\begin{aligned} & A R_{abcd} + B R_{adbc} + C g_{ab} R_{cd} + D g_{cd} R_{ab} + E g_{ac} R_{bd} + F g_{bd} R_{ac} \\ (43) \qquad & + H g_{ad} R_{bc} + I g_{bc} R_{ad} + J g_{ab} g_{cd} R + K g_{ac} g_{bd} R \\ & + L g_{ad} g_{bc} R + M g_{ab} g_{cd} + N g_{ac} g_{bd} + P g_{ad} g_{bc} = 0. \end{aligned}$$

The terms in (38) omitted in this equation depend on the ones kept, owing to the symmetry relations of R_{abcd}, and we use the customary notation for the tensors obtained from it by contraction.

As we may be able to factor out the fundamental quadratic tensor from (43), in case every term contains a g_{ab}, say, — i. e., if all the coefficients were zero except C, J, M, — the factoring being accomplished by contracting with respect to a and b, the equation may reduce to one of the second order, or to a scalar relation. We shall treat these simpler cases first. The scalar relation will evidently reduce to the form

$$(44) \qquad R = M,$$

M being a scalar, perhaps zero.

* Philosophical Magazine, vol. 45 (1923), p. 998 ff. Cf. H. Weyl, *Raum, Zeit, Materie*, fourth edition, p. 287.

If the equation reduces to one of the second order, it will be of the form

$$C R_{cd} + J g_{cd} R + M g_{cd} = 0. \tag{45}$$

On contracting this with respect to c and d, we obtain

$$C R + n J R + n M = 0, \tag{46}$$

n being the dimensionality of the space. If C is zero, this either gives an equation of type (44), having (45) as a consequence, or all the constants are zero and (45) is satisfied identically. If $C \neq 0$, on replacing $JR + M$ by $-C/n$ in (45), and dividing out C, we find

$$R_{cd} - \frac{1}{n} g_{cd} R = 0 \tag{47}$$

as the standard form for a tensor equation of the second order.

We may show that if, in (43), both A and B are zero, that equation is equivalent to one of type (44) or (47). For in that case, on contracting with respect to a and b, we would obtain

$$\begin{aligned} &(nC + E + F + H + I) R_{cd} \\ &\qquad + (D + nJ + K + L) g_{cd} R + (nM + N + P) g_{cd} = 0. \end{aligned} \tag{48}$$

If the coefficient of R_{cd} in (48) is different from zero, it leads to an equation of type (47), by means of which we may eliminate all the R's with two subscripts from (43), the resulting equation easily reducing to a relation like (44). As these two equations of our earlier types would have (43) as a consequence, nothing new results from this case. If the coefficient of R_{cd} in (48) were zero, we could contract (43) with respect to a second pair of subscripts and carry out the argument as before, unless all six coefficients were zero, in which case we would have $C = D = E = F = H = I = 0$, and (43) would be essentially a scalar equation like (44).

When A and B are not both zero, by using the relation

$$R_{abcd} + R_{adbc} + R_{acdb} = 0 \tag{49}$$

we can obtain a relation involving only one of these. For, since (43) is true for all coördinates, and hence in particular when we permute the subscripts, it gives, on interchanging b and c,

$$A R_{acbd} + B R_{adcb} + \cdots = 0, \tag{50}$$

or, using the symmetry properties of R_{abcd}.

$$(51) \qquad -A R_{acdb} - B R_{adbc} + \cdots = 0.$$

On subtracting this equation from (43), and using (49), we find

$$(52) \qquad (2B - A) R_{adbc} + \cdots = 0.$$

If the coefficient is zero, we interchange the rôles of A and B in the argument, and as they are not both zero, we see that in all cases an equation of form (43) is obtained containing a single R with four subscripts, with non-vanishing coefficient. Equation (52) may thus be solved for R_{adbc}, and if we use this value in the left member of the identity

$$(53) \qquad R_{badc} - R_{bacd} + R_{abcd} - R_{abdc} = 4 R_{abcd}$$

which follows from the symmetry properties of R_{abcd}, the resulting equation takes the form

$$(54) \qquad \begin{aligned} &A R_{abcd} + B(g_{ac} R_{bd} + g_{bd} R_{ac} - g_{ad} R_{bc} - g_{bc} R_{ad}) \\ &\qquad + C(g_{ac} g_{bd} - g_{ad} g_{bc}) R + D(g_{ac} g_{bd} - g_{ad} g_{bc}) = 0. \end{aligned}$$

Furthermore, (43) can not imply anything more than (54) unless perhaps equations of our earlier types, since we can eliminate all the terms with R's of four subscripts from (43) by using (54), coming back to the case where A and B in (43) are both zero.

On contracting (54) with respect to a and d, we find

$$(55) \quad (A + (2-n) B) R_{bc} + (-B + (1-n) C) g_{bc} R + D(1-n) g_{bc} = 0.$$

If the first coefficient here is not zero, we may express R_{bc} in terms of $g_{bc} R$ and g_{bc}, and consequently modify B at pleasure in (54), provided we make corresponding changes in C and D. Thus we may assume in all cases

$$(56) \qquad A + (2-n) B = 0.$$

Similarly, if the second coefficient is not zero, we may, by a second contraction, express R as a scalar, and hence change C in (49) by making the proper changes in D, so as to get in all cases

$$(57) \qquad -B + (1-n) C = 0.$$

Finally, when these changes have been made, if necessary, (55) gives by contraction

$$D(1-n) = 0. \tag{58}$$

Solving (56), (57) and (58) for the constants in terms of A, and then dividing out by A, which is not zero, we reduce (54) to the form

$$\begin{aligned} R_{abcd} &+ \frac{1}{n-2}(g_{ac}R_{bd} + g_{bd}R_{ac} - g_{ad}R_{bc} - g_{bc}R_{ad}) \\ &- \frac{1}{(n-2)(n-1)}(g_{ac}g_{bd} - g_{ad}g_{bc})R = 0. \end{aligned} \tag{59}$$

In case any of the transformations of constants were necessary to make the coefficients of (55) vanish, (54) implies, in addition to (59), equations of type (44) and (47), but nothing further in any case.

Before considering the possibilities of combining these three types of equation, we shall prove that an equation of type (47) always leads to a special case of one of type (44), that in which the right member is a constant.* If (47) holds, we have

$$R^b_{a/b} - \frac{1}{n}(Rg^b_a)_{/b} = 0, \tag{60}$$

denoting covariant derivatives in the usual way. On the other hand, it is well known, and easily proved by calculation in geodesic coördinates, that

$$R^b_{a/b} - \frac{1}{2}(Rg^b_a)_{/b} = 0, \tag{61}$$

identically, which shows that, unless $n = 2$,

$$(Rg^b_a)_{/b} = \partial R/\partial x_a = 0. \qquad R = R_0. \tag{62}$$

When $n = 2$, (47) holds identically, so that from now on we need only consider the combination of (62) and (47).

In view of the preceding results, we see that all sets of equations of the kind discussed in this section are equivalent to one of the following five tensor equations:

* G. Herglotz, Leipziger Berichte, vol. 68 (1916), p. 203; cf. also G. D. Birkhoff, loc. cit., p. 220.

$$\text{(I)}\quad R = M,$$

$$\text{(II)}\quad R_{ab} - \frac{1}{n} g_{ab} R_0 = 0,$$

$$\text{(III)}\quad R_{abcd} + \frac{1}{n-2}(g_{ac}R_{bd} + g_{bd}R_{ac} - g_{ad}R_{bc} - g_{bc}R_{ad})$$

$$- \frac{1}{(n-2)(n-1)}(g_{ac}g_{bd} - g_{ad}g_{bc})R = 0, \tag{63}$$

$$\text{(IV)} = \text{(III), (II)}\quad R_{abcd} + \frac{1}{n(n-1)}(g_{ac}g_{bd} - g_{ad}g_{bc})R_0 = 0,$$

$$\text{(V)} = \text{(III), (I)}\quad R_{abcd} + \frac{1}{n-2}(g_{ac}R_{bd} + g_{bd}R_{ac} - g_{ad}R_{bc} - g_{bc}R_{ad})$$

$$- \frac{1}{(n-2)(n-1)}(g_{ac}g_{bd} - g_{ad}g_{bc})M = 0.$$

Here, as indicated, (IV) is equivalent to (III) and (II); while (V) is equivalent to (III) and (I). It is evident that (IV) follows from (III) and (II), and to deduce these from (IV) we have merely to contract (IV), obtaining (II), which then enables us to reduce (IV) to (III). We may handle (V) similarly. Since we have shown that (II) always implies the special case of (I)

$$R = R_0, \tag{64}$$

these are all the combinations that we need consider. It should be particularly observed in the above that R_0 is a numerical constant, while M is a scalar point function. Thus there is always some value of M for which (I) holds — it only becomes a condition when the form of M is given.

The equations above given may be interpreted geometrically. (I) states the value of the total curvature, and evidently only derives special significance when M is specialized. (II) is the condition that the "principal directions" become indeterminate at every point of the space.* (III) is a necessary and sufficient condition ($n > 3$) that the space be conformally representable on

* L. P. Eisenhart, Proceedings of the National Academy of Sciences, vol. 8 (1922), p. 24.

a euclidean space.* Its left member occurs in various investigations on conformal representation, and has been called the "conform curvature". (IV) is the condition that the space be "spherical".†

As stated previously, the equations holding in space time which are the analogues of Laplace's equation in the Newtonian theory of gravitation for the theory of Einstein must be of the type under discussion. Thus they must be one of the equations given in (63). Since (I) is not restrictive enough, and (III) is too restrictive, it follows that the only possible equation is (II), that selected by Einstein.

Recapitulating the work of this section, we have proved

THEOREM III. *Every set of invariant equations formed by equating to zero expressions involving the fundamental quadratic tensor, g_{ij}, its first and second derivatives, and these last linearly, is equivalent to a tensor equation of one of the five types given in* (63) *above.*

* H. Weyl, Mathematische Zeitschrift, vol. 2 (1918), p. 404; J. A. Schouten, Mathematische Zeitschrift, vol. 11 (1921), pp. 58ff.; cf. also H. W. Brinkmann, Proceedings of the National Academy of Sciences, vol. 9 (1923), p. 1, p. 172.

† Schouten, loc. cit., p. 75.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.

SOME PROPERTIES OF SPHERICAL CURVES, WITH APPLICATIONS TO THE GYROSCOPE*

BY

O. D. KELLOGG

In his paper entitled *On the gyroscope*†, Professor Osgood has reduced to the utmost simplicity certain important aspects of the theory of the motion of a rigid body which is dynamically symmetric about an axis through the center of mass. The present study derives its inspiration from his article. A fundamental rôle is there played by the geodesic curvature of the trace on the unit sphere with center at the fixed point, or at the center of mass, of the point where this sphere is cut by the axis of symmetry, and this fact suggests that it might be desirable to have on hand more information about the intrinsic properties of spherical curves.

Accordingly, the first part of the present paper is devoted to the development of a number of such properties. The subject, however, is not without interest for its own sake, and suggests numerous extensions and related problems; for instance, a systematic study of the relationships of certain properties of the curvature as a function of the length of arc with the corresponding geometric properties of the curve. One such property which has already received a good deal of attention is the "four vertex theorem" (Vierscheitelsatz) for plane ovals‡. Among other questions that might be raised are such simple ones as the following: what functional character of the curvature, in addition to periodicity, insures a closed curve on the sphere? When is a plane, or spherical curve, asymptotic to some closed curve? What interesting comparison theorems are there for pairs of curves whose curvatures stand in simple relationships of equality or inequality? And there is the further question to be considered of curves on more general surfaces.

The second part of the paper makes application of the results of the first part to the theory of the gyroscope. While some apparently new facts are there brought to light, mention should be made of a new way of establishing

* Presented to the Society, April 28, 1923.

† These Transactions, vol. 23 (1922), pp. 240–264. This paper will be referred to hereafter by the initial O.

‡ See, for instance, Blaschke, *Vorlesungen über Differentialgeometrie*, vol. I, 1921, p. 16.

some classical results. It will be recalled* that certain theorems on the sense of precession of the gyroscope, and inequalities on the longitudinal motion of the spherical pendulum, have hitherto required the use of the theory of elliptic functions, or of Cauchy's integral theorem, for their establishment, while others depend simply on the appraisal (Abschätzung) of definite integrals. It turns out that *exactly those results which have previously required the less elementary methods are simple consequences of our present geometric theorems.*

In addition to the applications mentioned, there will be found in the second part of the paper certain further results and formulas which may well be of use in connection with problems on the gyroscope.

PART I. ON CERTAIN INTRINSIC PROPERTIES OF CURVES

1. Plane curves whose curvatures approach limits as the arc lengths increase indefinitely. The need of care in the study of qualitative properties of curves is illustrated by a loose statement in the authorized German translation of Cesàro's excellent book on intrinsic geometry.† In the discussion of plane curves the following sentences appear, in which ϱ is the radius of curvature, and φ the angle between a fixed line and the tangent at the point corresponding to the arc length s: "Nur wenn s zusammen mit φ unbegrenzt zunimmt, kann es vorkommen, daß ϱ sich einem von Null verschiedenen Grenzwerte a nähert. *Alsdann windet sich die Kurve, anstatt sich um einen Punkt herumzuwickeln, asymptotisch um einen Kreis vom Radius a, und zwar innerhalb oder außerhalb desselben, je nachdem der absolute Betrag von ϱ sich oberhalb oder unterhalb seines Grenzwertes hält.*" The following example shows that from the knowledge that ϱ approaches a limit as s becomes infinite we cannot infer that an asymptotic circle exists.

Let s_n denote the sum of the first n terms of the harmonic series, $s_n = 1 + 1/2 + 1/3 + \cdots + 1/n$. We describe about each point $(s_n, 0)$ of the x, y-plane, a circle of radius a. We then erase the upper halves of these circles, and unite the lower halves to a single continuous curve, with continuous curvature, by joining the two most distant extremities of each pair of successive semicircles by an arch of an ellipse with transverse axis along the x-axis, so chosen as to make the curvature continous. Thus, the first arch has its major axis terminating in the points $(1-a, 0)$ and $(3/2+a, 0)$, the second, in $(3/2-a, 0)$ and $(11/6+a, 0)$, and so on.

---

* See O, p. 260, end of page, and Appell, *Traité de Mécanique Rationnelle,* vol. I, Paris, 1902, p. 501.

† *Vorlesungen über natürliche Geometrie,* Leipzig, 1901, p. 12.

Here the curvature is seen to approach the limit $1/a$, but no asymptotic circle exists. The example can evidently be modified to bring out a number of different facts. For instance, the initial set of circles may be taken with their centers on a closed curve, say a large circle. We shall then have a bounded curve exhibiting the same lack of an asymptotic circle.

A statement that can be made, however, is the following:

THEOREM I. *If a plane curve has curvature, K, which is a continuous function of the arc length, s, and if, as s becomes infinite, K approaches a limit different from 0 while always increasing or always decreasing, then the curve has an asymptotic circle, approached from without, or within, respectively.*

As the proof of this theorem is analogous to that of a later one on spherical curves (p. 509), it will not be given. Instead, we shall prove a theorem which is essentially broader in its hypothesis, and whose analogue for spherical curves is neither so simple of treatment, nor so interesting from the point of view of dynamics:

THEOREM II. *If there exist two constants, $\alpha > 0$, and s_0, such that for $s_0 < s < \infty$, $K(s) > \alpha$, and if $K(s)$ is of bounded variation, then the curve defined by $K = K(s)$ has an asymptotic circle.*

A consideration of the total variation, $t(s)$, of $K(s)$, makes it obvious that $K(s)$ must approach a limit, k. Moreover, if $\tau(s)$ is the angle which the tangent to $\mathfrak{C}$, the curve under consideration, makes with a fixed direction, then K is also of bounded variation when considered a function of τ $\left(\tau(s) = \int_{s_0}^{s} K(s)\, ds\right)$. Now the coördinates, $x(s), y(s)$, of P, on $\mathfrak{C}$, referred to appropriate axes, are given by

$$x = \int_0^{\tau} \frac{\cos \tau}{K(\tau)}\, d\tau, \qquad y = \int_0^{\tau} \frac{\sin \tau}{K(\tau)}\, d\tau.$$

From these formulas it is not difficult to show that the successive maxima and minima of x and y approach limits, and that the mean of these limits for x, and the mean for y, give the coördinates of the center of an asymptotic circle.

The maxima, x_n'', of x occur for $\tau = (2n + \frac{1}{2})\pi$, and the minima, x_n', for $\tau = (2n + \frac{3}{2})\pi$. If the interval of integration in the expression for x_n'' be subdivided at these points, we may use the law of the mean in each sub-interval, and write

$$x_n'' = \int_0^{(2n+\frac{1}{2})\pi} \frac{\cos\tau}{K(\tau)}d\tau = \int_0^{\frac{\pi}{2}} \frac{\cos\tau}{K(\tau)}d\tau + \sum_{i=0}^{i=2n-1} \int_{(i+\frac{1}{2})\pi}^{(i+\frac{3}{2})\pi} \frac{\cos\tau}{K(\tau)}d\tau$$

$$= \frac{1}{K(\tau_0)} - \frac{2}{K(\tau_1)} + \frac{2}{K(\tau_2)} + \cdots + \frac{2}{K(\tau_{2n})}$$

$$= \frac{1}{K(\tau_0)} - 2\{[1/K(\tau_1) - 1/K(\tau_2)] + [1/K(\tau_3) - 1/K(\tau_4)]$$

$$+ \cdots + [1/K(\tau_{2n-1}) - 1/K(\tau_{2n})]\},$$

where $\tau_0, \tau_1, \ldots, \tau_{2n}$ are appropriate mean values of τ.

Now, since $K(\tau) > \alpha > 0$, $1/K(\tau)$ is of bounded variation with $K(\tau)$, for $|1/K(b) - 1/K(a)| < |K(b) - K(a)|/\alpha^2$. Hence the terms in the braces are the first n terms of an absolutely convergent series. Accordingly, x_n'' approaches a limit, x''. Similarly, x_n', y_n'' and y_n' approach limits, x', y'', and y'. Let us denote by a and b the means, $(x' + x'')/2$ and $(y' + y'')/2$, of these limits, and by r, the limit, $1/k$, of $1/K(\tau)$. Then, since $\int \frac{\cos\tau}{K(\tau)}d\tau$ and $\int \frac{\sin\tau}{K(\tau)}d\tau$, taken over any interval whose end points correspond to successive integral multiples of $\pi/2$, approach r, or $-r$, according to the interval selected, we see that the points of $\mathfrak{C}$ of maximum abscissas, maximum ordinates, minimum abscissas, and minimum ordinates, approach $(a + r, b)$, $(a, b + r)$, $(a - r, b)$ and $(a, b - r)$, respectively. Let τ_1 be a number, corresponding to a given $\varepsilon > 0$, such that for $\tau > \tau_1$, the differences between these four variables and their limits are numerically less than ε, and also that $|1/K(\tau) - 1/k| < \varepsilon$. This means that for $\tau > \tau_1$, the extremes of the coördinates of $\mathfrak{C}$ differ by less than ε from the corresponding extremes for the circle C: $X = a + r\sin\tau$, $Y = b - r\cos\tau$. From this we infer that for any point of $\mathfrak{C}$ for $\tau > \tau_1$,

$$|x(\tau) - X(\tau)| < \varepsilon + \int |[1/K(\tau) - 1/k]\cos\tau|\, d\tau < 2\varepsilon,$$

where the interval of integration is from the greatest multiple of $\pi/2$ less than τ, to τ. A similar inequality will hold for $y(\tau) - Y(\tau)$. Thus $\mathfrak{C}$ ultimately lies entirely in any region containing the circumference C in its interior; that is, $\mathfrak{C}$ approaches C in the weak sense. It is easily shown, however, that the approach has also the stronger sense, namely, that the difference of the direction angles, τ and τ' of $\mathfrak{C}$ and C, at two points, one on each curve, also

approaches 0 has τ increases indefinitely and the two points approach coincidence. This is because of the constant curvature of C, which has as consequence the finite distance apart of any pair of points at which the directions of C differ by a finite amount. The required inequalities may readily be supplied, and the proof of the theorem thus completed.

2. **Spherical curves and spherical evolutes.** The most convenient analytical tool for the present study appears to be the vector. We shall denote vectors by Clarendon letters, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{P}$, etc., and employ the Gibbs notation, $\mathbf{a}\cdot\mathbf{b}$ for the scalar product, $\mathbf{a}\times\mathbf{b}$ for the vector product, and $(\mathbf{a},\mathbf{b},\mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ for the triple product. The vector algebra required goes little beyond the distributive laws, and the expansion formula $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$. Primes will be used for one purpose only, namely, to denote derivatives with respect to the arc-length, s, of $\mathfrak{C}$. Unit vectors along the coördinate axes will be denoted by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and their senses we shall assume fixed once for all, and so related to the definition of vector product that $(\mathbf{i},\mathbf{j},\mathbf{k}) = +1$. The magnitude of a vector will be denoted by the corresponding italic letter.

Let the curve $\mathfrak{C}$ lie on the unit sphere, S, and let a variable point of $\mathfrak{C}$ be characterized by the unit *position vector*, $\mathbf{P} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, with origin at O, the center of S. We shall restrict ourselves to curves such that x, y and z have continuous derivatives with respect to s of the first three, and, in some cases, the first four orders. From $\mathbf{P}$ are derived the tangent vector

$$\mathbf{T} = \mathbf{P}' \tag{1}$$

and the curvature vector

$$\mathbf{K} = \mathbf{T}' = \mathbf{P}''. \tag{2}$$

For these we have the relations

$$\mathbf{P}^2 = 1, \tag{3}$$

since $\mathfrak{C}$ is on the unit sphere, and

$$\mathbf{T}^2 = 1, \tag{4}$$

since $(\mathbf{P}')^2 = (ds/ds)^2 = 1$.

Furthermore, by differentiating these relations, we find

$$\mathbf{P}\cdot\mathbf{T} = 0, \tag{5}$$

$$\mathbf{T}\cdot\mathbf{K} = 0, \tag{6}$$

and by further differentiation,

$$P\cdot K = -T^2 = -1, \tag{7}$$

$$P\cdot K' = -T\cdot K = 0, \tag{8}$$

$$T\cdot K' = -K^2. \tag{9}$$

The unit principal normal vector, N, of $\mathfrak{C}$ will coincide in direction with K We shall give it the same sense, so that

$$N = K/K = \varrho K, \tag{10}$$

where K and ϱ are the curvature and radius of curvature of $\mathfrak{C}$. Formula (7) shows that K always makes an obtuse angle with the position vector, and that its magnitude, the curvature, is never less than 1.

The unit binormal vector is defined as

$$B = T\times N. \tag{11}$$

If the initial point of B be placed at O, its tip will mark a point, Q, on S which is the *spherical center of curvature* of $\mathfrak{C}$ for the point P (the tip of P). We define the quantity R, $0 \leqq R \leqq \pi$, by the equations

$$\sin R = \varrho, \qquad \cos R = P\cdot B, \tag{12}$$

and call it the *spherical radius of curvature*. The same term will be used occasionally for the great circle arc, QP, of which it is the length. If, with Q as center, and with spherical radius R, a circle be described on the sphere, this will be the osculating circle of $\mathfrak{C}$ at P.

As P moves along $\mathfrak{C}$, Q will, in general, trace a curve, $\mathfrak{E}$, the *spherical evolute* of $\mathfrak{C}$. The point diametrically opposite Q will trace a symmetric curve, similarly related to $\mathfrak{C}$. But the particular evolute here defined is selected by the sense given to the binormal vector. A reversal of the sense of increasing s on $\mathfrak{C}$ would interchange the evolutes.

The Frenet formulas take the form

$$T' = \frac{N}{\varrho}, \qquad N' = \frac{B}{\tau} - \frac{T}{\varrho}, \qquad B' = -\frac{N}{\tau}, \tag{13}$$

where $1/\tau$ is the torsion of $\mathfrak{C}$. From the third Frenet formula, and equations (12), (10) and (7), we see that the torsion is the negative of the derivative with respect to s of the spherical radius of curvature,

$$R' = -\frac{1}{\tau}. \tag{14}$$

The torsion may be positive or negative; if the axis system is a right hand one, the curve, at a point where $\tau > 0$, deviates from its osculating circle by bending to the left as one follows $\mathfrak{C}$ in the sense of increasing s, with head in the direction in which P points.

The geodesic curvature, $\varkappa$, is simply related to R. The great circle tangent to $\mathfrak{C}$ at P has, as unit normal, the vector $P\times T$, and the absolute value of $\varkappa$ is the magnitude of the derivative with respect to s of this vector, that is, by (2), $\sqrt{(P\times K)^2}$. This reduces, with the help of (3) and (7), to $\sqrt{K^2-1}$. In terms of R, this is $|\cot R|$. It will be convenient to give $\varkappa$ a sign, so we shall identify it with $\cot R$*,

$$\varkappa = \cot R. \tag{15}$$

Thus, if a right hand system of axes is postulated, $\varkappa > 0$ when $\mathfrak{C}$ turns toward the left of its tangent great circle.

Combining (14) and (15), we have anew Professor Haskins' result (O, p. 248):

$$\frac{d}{ds}\tan^{-1}\varkappa = \frac{1}{\tau}.$$

3. Spherical curves. Osculating and asymptotic circles. For the purposes of applications to problems in dynamics, where coördinates are, in general, analytic functions of the time, and of arc length of paths, it will be adequate to consider curves whose curvatures are, in given intervals, either always increasing, or always decreasing functions of s. Accordingly we shall assume in this section, unless the contrary is stated, that in the open interval considered, $s_0 < s < s_1$, K' is either always positive, or else always negative. The fact that K is never negative permits the inequality $K' > 0$ to embrace two symmetric types of curves between which it is

* See O, p. 248. The a on the page referred to is the spherical radius of curvature of the intersection of the cone with the sphere.

desirable to distinguish. We have, indeed, $\varrho' < 0$, but the first equation (12) shows that $\cos RR' < 0$, so that $R' \gtrless 0$ according as $R \gtrless \pi/2$. A similar situation obtains with respect to $K' < 0$. Greater clarity will be attained when it is possible to think of the spherical center of curvature as the *nearer* of the two points which may play the rôle, so that in case $R > \pi/2$ for the arc $s_0 < s < s_1$, of $\mathfrak{C}$, we shall replace $\mathfrak{C}$ by the curve symmetric to it with respect to some diametral plane. The vectors $\boldsymbol{P}$, $\boldsymbol{T}$ and $\boldsymbol{K}$ will thus go over into symmetrically placed vectors, but $\boldsymbol{B}$ will go over into the symmetric vector with sense reversed, so that the sign of $\cos R$ will be changed. It will be noticed that the hypothesis that K' is always positive, or else always negative, on an arc precludes the possibility of an osculating great circle at an interior point of the arc.

The third Frenet formula (13) becomes, with the help of (14),

$$B' = R'N. \tag{16}$$

Accordingly, if σ represents the arc length of $\mathfrak{E}$, increasing with increasing s, the equality of magnitudes and directions in this vector equation yields the result, for the case $R' < 0$:

THEOREM III. *The tangent to the spherical evolute $\mathfrak{E}$ at Q has, for direction, the initial direction of the shorter great circle arc from Q to P, and the differential of arc of $\mathfrak{E}$ is given by*

$$d\sigma = -dR. \tag{17}$$

Thus, if a flexible inextensible string be unwound from a metal guide having the form of the curve $\mathfrak{E}$, the string being kept taut and in contact with the sphere, one of the points of the string will trace the curve $\mathfrak{C}$, so that in this sense the name *evolute* for $\mathfrak{E}$ is appropriate.

The following property of the evolute will also prove useful:

THEOREM IV. *No arc of $\mathfrak{E}$ on which R' is always positive, or always negative, is an arc of a great circle.*

Suppose, contrary to the theorem, that $\mathfrak{E}$, for $s' < s < s''$, is an arc of a great circle. Then, if $\boldsymbol{A}$ denote a constant vector perpendicular to the plane of this arc, we have $\boldsymbol{A}\cdot\boldsymbol{B} = 0$. If this equation be differentiated, the result reduces, by (16), to $\boldsymbol{A}\cdot\boldsymbol{N} = 0$, since $R' \neq 0$. But $\boldsymbol{A}\cdot\boldsymbol{N} = 0$ and $\boldsymbol{A}\cdot\boldsymbol{B} = 0$ imply that $\boldsymbol{T}$ is parallel with $\boldsymbol{A}$, a constant vector, which is impossible for a spherical curve.

We may now infer some properties of a given spherical curve, $\mathfrak{C}$. The first is

THEOREM V. *If $\mathfrak{C}$ is an arc of ever increasing curvature, $K' > 0$, $s' < s < s''$, any osculating circle $C_2(s = s_2)$ lies entirely within any osculating circle $C_1(s = s_1)$ for which $s' \leqq s_1 < s_2 \leqq s''$.* Here "within" means in that one of the two open regions, into which the circle divides the spherical surface, with the less area.

The theorem is an immediate consequence of equation (17), which, integrated from s_1 to s_2, yields $\sigma_{12} = R_1 - R_2$, where σ_{12} is the total length of $\mathfrak{C}$ between the points characterized by s_1 and s_2. Here the vital significance of the hypothesis of monotonic change in R (or K) appears, for σ is always changing in the same sense. Without the hypothesis, $\mathfrak{C}$ would have cusps, and σ_{12}, instead of representing the total length of the arc $\mathfrak{C}_{12}$, would give the algebraic sum of lengths between cusps.

Now R_1 and R_2 are less than $\pi/2$, and, therefore, so is σ_{12}. The latter, not being the measure of a great circle arc, by Theorem IV, must be greater than c_{12}, the length of the geodesic joining the ends of the arc $\mathfrak{C}_{12}$. Hence $c_{12} < R_1 - R_2$. But R_1 and R_2 are the spherical radii of the osculating circles C_1 and C_2, and c_{12} is the spherical distance between their centers, so that the inequality just derived is the necessary and sufficient condition that C_2 lie entirely within C_1.

A more general, though less intuitive, statement may be made with a different sense of "within" and the use of the geodesic curvature, so that osculating great circles are admitted. The curve $\mathfrak{C}$ must cross each of its osculating circles if the geodesic curvature has a derivative different from zero at the point of osculation. If "within" means in that region into which $\mathfrak{C}$ enters with increasing s, we may state the theorem: *if for $s' < s < s''$, $\varkappa'$ is either always positive or always negative, the later osculating circles always lie within the earlier ones.* It will be seen how this follows from the theorem as first stated when it is observed that the arc consists at most of two pieces, on one of which $K' > 0$, and on the other of which $K' < 0$.

An immediate corollary of Theorem V is

THEOREM VI. *Under the hypothesis of Theorem* V, *the arc $s_1 < s < s_2$ of $\mathfrak{C}$ lies entirely within C_1 and entirely without C_2.* More generally, *an arc of $\mathfrak{C}$ on which $\varkappa'$ is always positive, or always negative, lies to one side of its osculating circle at either extremity of the arc.*

For if $\mathfrak{C}$ cut an osculating circle, or touched it, at a point other than the point of osculation, there would exist two osculating circles with a common point. This is contrary to Theorem V.

One more step yields the analogue for spherical curves of Theorem 1:

THEOREM VII. *If $\mathfrak{C}$ be supposed to be infinitely long, and if, from some point on, $K' > 0$, $\mathfrak{C}$ approaches an asymptotic circle or point, according as K is bounded or not. If $K' < 0$, $\mathfrak{C}$ approaches an asymptotic circle.*

We may assume $R < \pi/2$, as indicated previously. Then as s increases indefinitely, R, whose sine is $1/K$, and which therefore changes monotonely, must approach a limit, a, $0 \leq a \leq \pi/2$. Accordingly, by (17), σ approaches a limit. Hence the center of spherical curvature, Q, of $\mathfrak{C}$ approaches a limiting position, and as R approaches a, it follows that $\mathfrak{C}$ approaches the asymptotic circle which has for center the limiting position of Q, and for radius, a. This circle may reduce to a point if K is increasing, or to a great circle if K is decreasing.

We close this section with the establishment of one more fact, which we shall need in what follows.

THEOREM VIII. *If the projections, x, y, z, of P have four continuous derivatives with respect to s, $\mathfrak{C}$ bends, with increasing s, to the same or the opposite side of its tangent great circle as $\mathfrak{C}$, according as R is less than or greater than $\pi/2$.*

To see this, we compare the triple products (B, B', B'') and (P, P', P''), by expressing them in terms of the orthogonal set, T, N, B. We start by differentiating the third Frenet formula, $B' = -N/\tau$, and simplifying by means of the second:

$$B'' = \frac{N\tau'}{\tau^2} - \frac{N'}{\tau} = \frac{N\tau'}{\tau^2} - \frac{B}{\tau^2} + \frac{T}{\varrho\tau},$$

so that

$$(B, B', B'') = \left(B, -\frac{N}{\tau}, \frac{T}{\varrho\tau}\right) = \frac{1}{\varrho\tau^2}(T, N, B) = \frac{1}{\varrho\tau^2}.$$

On the other hand, $(P, P', P'') = (P, T, K) = (P, T, N)/\varrho$, by (1), (2), and (10). But P must be expressed in terms of T, N, and B: $P = (P\cdot T)T + (P\cdot N)N + (P\cdot B)B$, or, by (5), (10), (7) and (12),

$$P = -\varrho N + \cos R\, B. \tag{18}$$

Hence $(P, P', P'') = \cos R/\varrho$, and we have, finally, $\tau^2 \cos R\,(B, B', B'') = (P, P', P'')$, an equation of which the theorem is a qualitative translation into words. It will be noticed that the hypothesis of differentiability precludes the vanishing of the denominators ϱ and τ in the above reasoning.

4. **Curves with monotonic co-latitude.** In a number of dynamical problems connected with the sphere, the motion is limited by two parallel circles. This section will be devoted to a curve, $\mathfrak{C}$, on the sphere, bounded by two such circles, the distance of $\mathfrak{C}$ from the plane of one of the circles being

ever increasing, or ever decreasing, and its spherical radius of curvature having the same property. We shall suppose that this curve passes from tangency to C_0 at P_0 for $s = s_0$ to tangency to C_1 at P_1 for $s = s_1$ $(s_1 > s_0)$. The subscripts 0 and 1 will be used generally to distinguish quantities or points connected with the beginning and end, respectively, of this arc of $\mathfrak{C}$. The direction of the common axis of C_0 and C_1, with the sense from the plane of C_0 to that of C_1, we shall call *north,* and take it for that of the z-axis. The x-axis of our orthogonal axis-system we take in the *prime meridian,* through P_0. The y-axis we take so as to form with the others a right hand system, or *eastward* as seen by an observer at P_0. For the sake of definiteness we shall suppose that $\mathfrak{C}$ runs initially eastward, a restriction to be removed later.

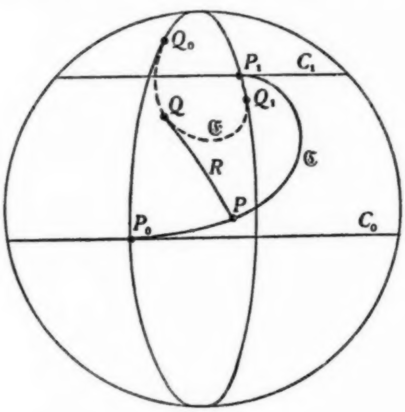

Fig. 1.

THEOREM IX. *Let $\mathfrak{C}$ be a curve on the unit sphere, S, whose position vector, $\mathbf{P}$, has four continuous derivatives with respect to s on the closed interval $s_0 \leqq s \leqq s_1$, and let C_0 and C_1 denote two parallel circles on S, neither of them point circles. Let $\mathfrak{C}$ pass from tangency to C_0 to tangency to C_1 running initially eastward, under the following conditions on the open interval $s_0 < s < s_1$:*
(*a*) *the point P, of $\mathfrak{C}$, corresponding to the arc-length s has ever increasing distance from the plane of C_0: $z' > 0$,*
(*b*) *the spherical radius of curvature, R, of $\mathfrak{C}$, is ever decreasing: $R' < 0$.*

Then the longitude of P_1, measured positively to the east, exceeds the longitude of P_0.

It will be noticed that the hypothesis does not eliminate the possibility of an osculating great circle, for R may exceed $\pi/2$. It is for this reason that the spherical radius of curvature is more appropriate for the present purpose

than the curvature, K. The particular evolute here used is that defined previously (p. 506).

An intuitive notion of the proof of the theorem may be gained from Figure 1. We compare the longitude, ψ, of P, on $\mathfrak{C}$, with the longitude, φ, of Q, on $\mathfrak{E}$. Initially $\psi = \varphi = 0$, but ψ increases initially more rapidly than φ, and remains greater, while φ is always increasing. Hence ψ is positive at P_1. What follows is merely an examination of the details.

We begin with an analytic formulation of the hypotheses:

$$
\begin{aligned}
&\text{for } s = s_0,\ s = s_1, && \boldsymbol{T}\cdot\boldsymbol{k} = 0,\\
&\text{for } s = s_0, && \boldsymbol{T} = \boldsymbol{j},\\
(19)\qquad &\text{for } s_0 < s < s_1, && R' < 0,\\
& && -1 < z_0 < \boldsymbol{P}\cdot\boldsymbol{k} < z_1 < +1,\\
& && (\boldsymbol{P}\cdot\boldsymbol{k})' = \boldsymbol{T}\cdot\boldsymbol{k} > 0.
\end{aligned}
$$

The identity connecting the direction cosines of $\boldsymbol{k}$ with respect to the orthogonal set $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}$ will also prove useful:

$$
(20)\qquad (\boldsymbol{T}\cdot\boldsymbol{k})^2 + (\boldsymbol{N}\cdot\boldsymbol{k})^2 + (\boldsymbol{B}\cdot\boldsymbol{k})^2 = 1.
$$

Finally, it will be convenient to employ two vectors in the equatorial plane with the same longitudes as P and Q,

$$
(21)\qquad \boldsymbol{p} = \frac{\boldsymbol{P}-(\boldsymbol{P}\cdot\boldsymbol{k})\boldsymbol{k}}{\sqrt{1-(\boldsymbol{P}\cdot\boldsymbol{k})^2}}, \qquad \boldsymbol{b} = \frac{\boldsymbol{B}-(\boldsymbol{B}\cdot\boldsymbol{k})\boldsymbol{k}}{\sqrt{1-(\boldsymbol{B}\cdot\boldsymbol{k})^2}},
$$

in which the denominators do not vanish for $s_0 < s < s_1$ because of the relations (19_4), and (19_5), respectively, with (20).

Our first task is to see that the initial longitudes of P and Q may be taken as 0. The first, $\psi_0 = 0$, is a matter of definition, and is implied in the term "prime meridian". As Q is always in the plane through P normal to $\mathfrak{C}$, Q_0 is in the same meridian plane as P_0, and, by the definition of $\mathfrak{E}$, lies to the north of P_0. But the important point is that Q_0 is not separated from P_0 by the pole (the north pole is meant here and in what follows) on the shorter meridian arc connecting these points. This is because the osculating circle, C', of $\mathfrak{C}$ at P_0 cannot go south of C_0 without carrying $\mathfrak{C}$ with it, which is

contrary to hypothesis (19_4). Hence Q_0 is either at the pole, or on the prime meridian. In the latter case, its longitude is an integral multiple of 2π, which may be taken as 0. If Q_0 is at the pole, i. e. if C' coincides with C_0, we shall define φ_0 as the limit of φ as $s \to s_0$. It will presently appear that this limit exists, and is 0.

We next show that φ is always increasing. To do this, we recall that the magnitude of the derivative of a unit vector is the rate at which it is changing direction, i. e. that $|\varphi'| = \sqrt{(b')^2}$. A little reckoning is required to compute this quantity, but it is straight-forward, and need not be set down. One starts with (21_2) and uses the relations (1), (16) and (20), obtaining

$$\varphi' = \frac{-R'(T\cdot k)}{1-(B\cdot k)^2}, \tag{22}$$

in which the sign is determined as follows. Because of (19_1) and (19_3), φ' never vanishes for $s_0 < s < s_1$, and so, being continuous, keeps its sign. By Theorem VIII, $\mathfrak{E}$ bends in the same sense as $\mathfrak{C}$, or in the sense opposite to that of $\mathfrak{C}$, according as $R \gtrless \pi/2$; hence $\mathfrak{E}$ bends to the *left*. Moreover, by (16), $\mathfrak{E}$ runs initially southward, being tangent to the prime meridian, so that it bends to the east, and φ is initially increasing. The equation (22) then shows that φ' is always positive, and φ always increasing, as stated. The same considerations show that φ approaches 0 as $s \to s_0$, even if Q_0 is the pole.

It remains to show that $\psi - \varphi > 0$. To do this, we note that by the definition of vector product, $b \times p = k \sin(\psi - \varphi)$. Hence $\sin(\psi - \varphi) = (b, p, k)$. Another brief reckoning, involving (20), (18), and the fact that $B \times N = -T$, gives the result

$$\sin(\psi - \varphi) = \frac{\varrho(T\cdot k)}{\sqrt{1-(P\cdot k)^2}\sqrt{1-(B\cdot k)^2}}. \tag{23}$$

This shows that the angle $\psi - \varphi$, which is continuous, and which starts at 0, has a positive sine until P_1 is reached, so that $\psi_1 \geqq \varphi_1 > 0$, as was to be proved.

More, however, may be inferred from the above developments. The formula (23) shows that as $s \to s_1$, $\psi - \varphi \to 0$ or π. Now, since $R < \pi$, the latter case can occur only when the terminal spherical radius of curvature, which must lie along a meridian, lies across, or terminates in, the pole. This means that the final osculating circle, C'', contains C_1 (within that region in which Q_0 lies), or coincides with C_1. Also, $\psi - \varphi$ increases toward π, and φ

is always increasing, so that ψ is increasing at P_1. The result is that for this case $\mathfrak{C}$ touches C_1 still running eastward. If $\psi - \varphi \to 0$, as $s \to s_1$, the terminal radius of curvature cannot lie across the pole, and C'', with spherical radius less than the polar arc to P_1, must lie south of C_1, and $\mathfrak{C}$ then touches C_1 running west. We accordingly infer

THEOREM X. *If to the hypotheses of Theorem* IX, *we add* (*c*) $\mathfrak{C}$ *touches both circles* C_0 *and* C_1 *running eastward, then the difference in longitude of its points of contact with these circles exceeds* π.

It remains to consider the case $R' > 0$ (see Figure 2). As before, we may take $\psi_0 = 0$, and Q_0 is certainly between P_0 and the pole. For the former argument

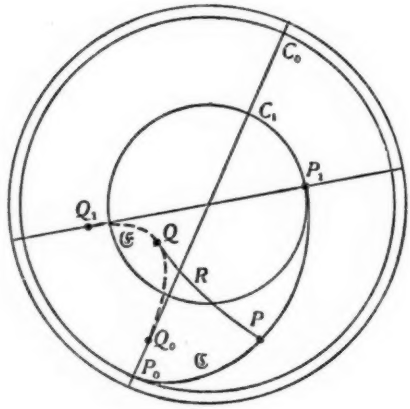

Fig. 2.

shows that Q_0 is not beyond the pole; and no more can Q_0 coincide with the pole, for from (16) we infer that $-\operatorname{cosec}^2 R R' = \varkappa'$, so that $\varkappa'$ would be negative, i. e. $\varkappa$ decreasing, and $\mathfrak{C}$ would have to bend to the south of C_0, contrary to hypothesis. As to $\mathfrak{C}$, (16) now shows that it runs northward, and hence to the west, by Theorem VIII. As $R' > 0$, formula (22) is valid without change, and φ, initially 0, continually decreases. But by (23), $\sin(\psi - \varphi)$ is positive, and it approaches 0 as $s \to s_1$.

If $\psi - \varphi \to 0$, the terminal spherical radius of curvature cannot cross the pole, so that the spherical radius of C'' is less than that of C_1. Hence, as C'' cannot lie north of C_1, its center also lies south of C_1. As φ is decreasing, and $\psi - \varphi$ approaching 0 through positive values, we infer that ψ is decreasing, and that in this case $\mathfrak{C}$ touches C_1 running west. We have here, $\psi_1 \leqq \varphi_1 < 0$, and the result is merely Theorem IX with the rôles of C_0 and C_1 interchanged, and $\mathfrak{C}$ replaced by its reflection in the prime meridian.

But if $\psi - \varphi \to \pi$, the terminal radius of curvature lies across the pole, and we infer that $\mathfrak{C}$ is still running eastward at P_1, so that we may state

THEOREM XI. *If in Theorem* IX *we replace the condition* (*b*) *by*
(*b'*) *the spherical radius of curvature of* $\mathfrak{C}$ *is always decreasing,* $R' < 0$,
and add the condition (*c*) *of Theorem* X, *it then follows that the difference in longitude of the points of contact of* $\mathfrak{C}$ *with the circles* C_0 *and* C_1 *is less than* π.

Finally, we consider the removal of the restriction that $\mathfrak{C}$ run initially eastward. If $\mathfrak{C}$ runs initially westward, we may consider its reflection in the plane of the prime meridian. The proofs of the corresponding theorems remain the same, except that the other evolute from the one we have been employing must be used. Or, we may use a left hand system of axes, which will produce the same effect. The three last theorems may now be stated as follows.

Let $\mathfrak{C}$ *run from tangency at* P_0 *to* C_0, *to tangency at* P_1 *to* C_1, *the distance of the moving point, P, on* $\mathfrak{C}$, *from the plane of* C_0 *having a positive derivative with respect to the arc length s for* $s_0 < s < s_1$. *Let R be the spherical radius of curvature of* $\mathfrak{C}$ *which is measured initially toward* P_0 *from the side of* C_0 *on which* C_1 *lies. Then*

(1) *if* $R' < 0$, *or if* $R' > 0$ *for* $s_0 < s < s_1$, *that one of the points* P_0, P_1, *for which R is the less, has the greater longitude, measured in the sense of increasing arc length at* P_0;

(2) *if* $R' < 0$ *for* $s_0 < s < s_1$, *and the longitude of P is changing in the same sense at* P_0 *and at* P_1, *the difference in longitude of these two points exceeds* π;

(3) *if* $R' > 0$ *for* $s_0 < s < s_1$, *and the longitude of P is changing in the same sense at* P_0 *and at* P_1, *the difference in longitude of these two points is numerically less than* π.

In the above statements, the geodesic curvature, $\varkappa$, might have been used instead of R, but it would have necessitated a subdivision of cases according as $\varkappa \gtrless 0$, or else a modification of the convention as to the sign of $\varkappa$. Neither seems desirable.

PART II. SOME POINTS IN THE THEORY OF THE TOP

5. Osgood's intrinsic equations. Notation. Some familiarity on the part of the reader with Professor Osgood's paper will be assumed in what follows. His intrinsic equations, however, will appear here with a change in a sign, and a slight change in notation.* We write them as follows:

* The change in sign is due to a different convention as to the senses in which certain quantities are measured. In justification of a departure from Professor Osgood's well considered conventions, I can only plead that I have felt surer footed in employing conventions to which I have been accustomed. Since there is little agreement in the literature on the subject, the departure will cause the reader little inconvenience.

$$A v \frac{dv}{ds} = T,$$

(24) $$A v^2 \varkappa - C r v = N,$$

$$C \frac{dr}{ds} = S.$$

Here C and A are the moments of inertia of the top about its axis of dynamic symmetry, and a perpendicular to the axis through the fixed point, respectively. The magnitude of the velocity of the point P, where the axis of the top (on which a positive sense has been arbitrarily fixed) pierces the unit sphere, is denoted by v; the geodesic curvature of $\mathfrak{C}$, the path of P, or the "*bending*" of the cone swept out by the axis of the top, is denoted by $\varkappa$; $\varkappa$ is positive when $\mathfrak{C}$ swerves to the left of an observer walking on the sphere and following the top axis. The sense of the positive unit tangent vector, $\boldsymbol{t}$, to $\mathfrak{C}$, is that of the motion, and that of the normal vector, $\boldsymbol{n}$ (here tangent to the sphere, and not to be confused with the principal normal vector, $\boldsymbol{N}$, of Part I), is to the left of the above mentioned observer. Thus a curve with positive $\varkappa$ bends toward the positive normal. The component of the applied moment, along the axis, or the *spin moment*, is denoted by S. The remaining component of the applied moment may be regarded as due to a force tangent to the sphere and applied at P. We denote the components of this force in the directions of $\boldsymbol{t}$ and $\boldsymbol{n}$ by T and N, respectively. In Professor Osgood's notation, this force has components T and Q along the tangent and negative normal vectors associated with $\mathfrak{C}$, but his normal vector points to the right, so that his Q and the present N are identical. Here, a positive moment or rotation about an axis would force a right hand screw forward in the positive sense along the axis. Thus our S and r are opposite in sign to the corresponding N and r of Professor Osgood's paper. The only change in sign resulting from these differences is in that of the term $C r v$ in the second intrinsic equation.

6. **The energy integral.** This is obtained, in case it exists, from equations (24_1) and (24_3), and may be written

(25) $$\frac{1}{2}(A v^2 + C r^2) = \int \left(T + S v \frac{dr}{ds}\right) ds + h.$$

provided the integrand is the derivative with respect to s of a function of positiòn, as it is, for instance, in the case of the heavy frictionless top with fixed peg (O, p. 256, (i)).

7. The heavy top with fixed peg. Initial sense of change of latitude. The equations of the motion may be written (O. p. 258, (6) and (iv), p. 256, (i))

$$
(26)\qquad
\begin{aligned}
\left(\frac{du}{dt}\right)^2 &= P(u),\\
\frac{d\psi}{ds} &= \frac{\varepsilon + \gamma(u_0 - u)}{r(1-u^2)},\\
\varkappa &= \frac{\gamma}{r^3}\left\{v_0^2 - \frac{a\varepsilon}{2\gamma} + \frac{a}{2}(u_0 - u)\right\},\\
r^2 &= v_0^2 + a(u_0 - u).
\end{aligned}
$$

Here $u = \cos\theta$, θ being the co-latitude of P; ψ is the longitude of P, positive when measured eastward; u_0, ψ_0, $\dot\psi_0$, and v_0, are initial values, corresponding to some moment when $\dot\theta\,(= d\theta/dt) = 0$; $\varepsilon = (1 - u_0^2)\,\dot\psi_0$; $v_0^2 = \sin^2\theta_0\,\dot\psi_0^2 = (1 - u_0^2)\,\dot\psi_0^2$; ν is the constant value of r; $\gamma = C\nu/A$;* $a = 2Mgh/A$, where M is the mass, and h the distance of the center of mass from the fixed point. The function $P(u)$ is given by

$$
(27)\qquad P(u) = (1 - u^2)\,[v_0^2 + a(u_0 - u)] - [\varepsilon + \gamma(u_0 - u)]^2.
$$

The motion takes place between two parallel circles, C_0 and C_1, which may, in special cases, reduce, one or both, to point circles. We shall disregard such special cases in the present study.

The first question that arises when the initial conditions include $\dot\theta_0 = 0$, as at present, is, does the top begin to rise, or to fall, or, is the initial limiting circle the lower, C_0, or the upper, C_1? To answer this, one compares the initial value of $\varkappa$, or

$$
(28)\qquad \varkappa_0 = \frac{1 - u_0^2}{v_0^3}\,[\gamma\,\dot\psi_0^2 - \frac{a}{2}\,\dot\psi_0],
$$

* We shall assume ν, and hence γ, to be positive, or when so specified, zero; never negative. It will be recognized that this restriction merely involves, in some cases, reflecting the motion in a mirror.

with the geodesic curvature, $\varkappa_c$, of the initial limiting circle. But $\varkappa_c = \operatorname{sgn} \dot\psi_0 \cot\theta = \operatorname{sgn} \dot\psi_0 \frac{u_0}{\sqrt{1-u_0^2}}$, or, as $v_0 = \sqrt{1-u_0^2} \operatorname{sgn} \dot\psi_0 \, \dot\psi_0$, $\varkappa_c v_0^3 = u_0(1-u_0^2)\dot\psi_0^3$.* Hence

$$(\varkappa_0 - \varkappa_c)\dot\psi_0 = -\frac{1}{v_0}\left[u_0 \dot\psi_0^2 - \gamma \dot\psi_0 + \frac{a}{2}\right]. \tag{29}$$

If $\varkappa_0 > \varkappa_c$, $\mathfrak{C}$ bends more rapidly to the left than the initial limiting circle, i. e. if running east, $\mathfrak{C}$ rises, and similarly for the other cases, the sign of $(\varkappa_0 - \varkappa_c)\dot\psi_0$ being decisive. The function in brackets is the quadratic whose roots give the longitudinal velocities of steady precession in the circle of latitude $\theta = \theta_0$. If $u_0 > 0$, $(\varkappa_0 - \varkappa_c)\dot\psi_0$ is positive between the roots of the quadratic, and if $u_0 < 0$, the opposite is the case. If $u_0 = 0$, there is but one root, and $(\varkappa_0 - \varkappa_c)\dot\psi_0 > 0$ for $\dot\psi_0$ greater than this root. Hence we may state

THEOREM XII. *If, at any instant, the path of P touches a circle of latitude, it will curve away from, or toward, the equator according as its longitudinal velocity at the instant does, or does not, lie between the longitudinal velocities for steady precession in that circle. If the circle is the equator, the path curves northward or southward according as the longitudinal velocity does, or does not exceed the single longitudinal velocity for steady precession on the equator. If the spin is so slight that the precessional values are coincident, or imaginary, the top always falls.*

It is, of course, understood that the initial longitudinal velocity is not a precessional value in the statement of the above theorem.

8. **Reversals in sense of the longitudinal motion.** Equation (26_2) gives the value of u for which the longitudinal motion changes sense: $u_l = u_0 + \varepsilon/\gamma$. In order, however, that this value of u shall correspond to a real point in the motion, it is necessary first, that it lie in the closed interval $(-1, +1)$, and secondly, that $P(u_l) \geqq 0$. As $P(u) < 0$ if $u < -1$, these conditions may be stated as follows:

$$\dot\psi_0 \leqq \frac{\gamma}{1+u_0}.$$

(30)

$$P(u_l) = \frac{(1-u_0^2)^3}{\gamma^2}\dot\psi_0\left(\dot\psi_0 + \frac{\gamma}{1+u_0}\right)\left(\frac{\gamma}{1-u_0} - \dot\psi_0\right)(\dot\psi_0 - 2\delta) \geq 0,$$

* The function $\operatorname{sgn} z$ is the familiar function *signum* z, defined as follows: for $z<0$, $\operatorname{sgn} z = -1$, for $z = 0$, $\operatorname{sgn} z = 0$, and for $z>0$, $\operatorname{sgn} z = +1$.

where $\delta = a/2\gamma$ is the value of the longitudinal velocity for steady precession on the equator.

9. **Points of inflection.** The curve $\mathfrak{C}$ has a point of inflection at P when its projection on the plane tangent to the sphere at P has a point of inflection at P, or, when $\mathfrak{C}$ crosses an osculating great circle. A necessary condition for this is $\varkappa = 0$; it is sufficient that $\varkappa$ change signs. Equation (26_3) gives, as the only value of u for which this can occur, $u_i = u_0 + (2v_0^2/a) - (\varepsilon/\gamma)$. Here, the conditions that u_i correspond to a real point in the motion may be given the form

$$F_1(\dot\psi_0) = \dot\psi_0^2 - \delta\,\dot\psi_0 - \frac{a}{2(1+u_0)} \leqq 0,$$

$$(31)\qquad P(u_i) = \frac{4(1-u_0^2)^3}{a^2}\,\dot\psi_0(\dot\psi_0-\delta)(\dot\psi_0-2\delta)$$
$$\times\left[(\dot\psi_0-\delta)\left(\dot\psi_0^2-\frac{\gamma^2}{1-u_0^2}\right)+\frac{a u_0}{1-u_0^2}\,\dot\psi_0\right] \geqq 0.$$

The factorization in the value of $P(u_i)$ has been carried as far as possible in the domain of rationality $(a, \gamma, u_0, \sqrt{1-u_0^2})$. This may be verified by considering the special rational values $a = 16/5$, $\gamma = 4/5$, $\delta = 2$, $u_0 = 3/5$, which reduce the cubic factor to the form $x^3 - 2x^2 + 2x + 2$, a polynomial evidently without rational roots.

10. **Monotonic curvature.** For the application of some theorems of Part I, we need assurance that $\varkappa'$ keeps its sign between the limiting circles. This derivative, obtained from (26_3), may be reduced, without excessive trouble, to

$$(32)\qquad \varkappa' = \frac{a\gamma}{r^5}\,u'\left[v_0^2 - \frac{3a\varepsilon}{4\gamma} + \frac{a}{4}(u_0 - u)\right].$$

Striking is the fact that like the curvature, $\varkappa$, and the longitudinal velocity, $\dot\psi$, this derivative is dependent for its sign on a *linear* function of u. The derivative, u', of u, keeps its sign between the limiting circles. The value of u for which $\varkappa'$ vanishes is $u_k = u_0 - (3\varepsilon/\gamma) + (4v_0^2/a)$, and the conditions that it characterize a real point in the motion may be given the form

$$F_2(\dot\psi_0) = \dot\psi_0^2 - \frac{3\delta}{2}\dot\psi_0 - \frac{a}{4(1+u_0)} \leq 0,$$

$$(33) \quad P(u_k) = \frac{48}{a^2}(1-u_0^2)^3\dot\psi_0\left(\dot\psi_0 - \frac{3}{2}\delta\right)(\dot\psi_0 - 2\delta)$$

$$\times\left[\dot\psi_0^2\left(\dot\psi_0 - \frac{3}{2}\delta\right) - \frac{a(a-6\delta^2 u_0)}{12\delta^2(1-u_0^2)}\left(\dot\psi_0 - \frac{a\delta}{2(a-6\delta^2 u_0)}\right)\right] > 0.$$

The cubic factor is, in this case also, irreducible in the domain $(a, \gamma, u_0, \sqrt{1-u_0^2})$, as may be seen by using the special values $a = 24$, $\delta = 2$, $u_0 = 0$.

11. **Applications of the geometry of spherical curves.** It is not the purpose of the present paper to enter into a detailed discussion of the various cases of motion of the top which may present themselves, although the materials gathered above permit new distinctions between types.* We shall rather content ourselves with the enunciation of certain typical results, following the customary cases (O, p. 258).†

Case I. The longitude is always increasing (O, Figure 3, I). In this case, it is usually assumed that the path of P has points of inflection. The illustrations of the most current text books show such a curve.‡ It should be noticed, however that such is not necessarily the case, and that in the present type of motion *both paths with inflections and paths without inflections can occur.* Thus, for $a = 2$, $\delta = \gamma = 1$, $u_0 = 1/2$, with $\dot\psi_0$ near to 1, the condition (30_1) for a change in sense of the longitudinal motion is not fulfilled, so that the longitude is always increasing. The condition (31_1) for a real point of inflection is fulfilled, while the condition (31_2) takes the form $\dot\psi_0 - 1 \leq 0$ for $\dot\psi_0$ near 1. Thus, with $\dot\psi_0$ slightly less than 1, we have a path with a point of inflection on each arc between the limiting circles, and with $\dot\psi_0$ slightly greather than 1, we have an inflectionless path.

If the spin be stopped $(\gamma = 0)$, the top becomes a spherical pendulum. The longitude always changes monotonely, (26_2), and $\varkappa$ reduces, by (26_3) to $-a\varepsilon/2v^3$. Thus *the path of the spherical pendulum never has inflection points.* The use of the intrinsic equations has rendered extremely simple the proof of a well known fact.

* Mr. A. H. Copeland, of the Graduate School at Harvard, is undertaking such a discussion in connection with his candidacy for the doctorate.

† Case I is the wavy curve without cusps or double points, Case II is the curve with cusps, and Case III, the curve with loops.

‡ Professor Osgood has also overlooked the necessity of imposing the conditions (31) on $\dot\psi_0$ in order to secure a path with inflection points (see O, p. 259).

Another fact about the spinning top that seems hitherto to have escaped explicit mention is the following: *there exist cases in which the longitude of P increases by more than π between two successive contacts with the limiting circles.**

Sufficient conditions for this type of motion are (see Theorem X): the longitude is always increasing, the top is rising from $u = u_0$, and $R' < 0$ (the definitions of R in Section 2 and Section 4 coincide in the present case, and we find from (15) that the last condition is equivalent to $\varkappa' > 0$). An example showing the compatibility of these conditions is the following: $a = 1$, $\gamma = 1$, $u_0 = 0.4$, $\dot{\psi}_0 = 1.44312$. We find $u_1 = 0.5$, $\varepsilon = 1.21222$, $v_0^2 = 1.74938$. The condition (30_1) for a change in sense of the longitudinal motion is contradicted, while $\dot{\psi}$ is initially positive; $\varkappa'$ is seen from (32) to be initially positive, while the value u_k (p. 519) for which $\varkappa'$ changes sign is found to be greater than 1, and so does not correspond to a real point on the path.†

Case II. Here $\dot{\psi}_0 = 0$, and by (32), $\varkappa'$ can vanish only for $u = u_0$. We infer, from Theorem VI, that *an arc of the path of P between two successive contacts with the limiting circles lies entirely within the osculating circle at the extremity of the arc at which the curvature is finite.*

Case III. The same situation obtains here, where the path has loops. Let us use $\mathfrak{C}$ to denote an arc of the path between the limiting circles. Then $\mathfrak{C}$ *lies entirely within its osculating circle at one extremity, and entirely without its osculating circle at the other extremity, one of these circles containing the other in its interior*, by Theorems V and VI. "Interior" may here be interpreted in the narrower sense, namely, the region with the less area.

To justify these statements with regard to Case III, we must show that for the looped curve, neither $\varkappa'$ nor $\varkappa$ vanishes *between* the limiting circles, i. e. that the conditions (33) and (31) are both incompatible with the conditions (30), when the strict *inequalities* are employed in the first two. We shall give the proof for the conditions (33), that for (31) being similar, and simpler.

In the first place, it is no restriction to assume that $\dot{\psi}_0 > 0$, for this merely means that a proper choice has been made of that limiting circle which is to be the initial one, inasmuch as $\mathfrak{C}$ meets the limiting circles running in

* If $P(u)$ has $u = 1$ as a double root, $\mathfrak{C}$ makes a spiral around the north pole. It seems entirely plausible that near this motion are others of the type in question, where the increase in longitude is arbitrarily large. It may also be of interest to note that a similar situation exists in a very elementary problem, namely, the following: a bead is free to slide under gravity without friction on a circular wire, which rotates with constant angular velocity about a vertical diameter. It will be found that the wire makes more than a half revolution between two successive times when the bead attains its extreme heights.

† A computation, which I believe to be accurate, gives, to four significant figures, $\Psi_{01} = 3.428$, which is about 9.1 percent in excess of π.

opposite senses. This assumption greatly simplifies (30_2), reducing it to $\dot\psi_0 - 2\delta > 0$, so that the conditions for the loopy type of curve become

$$2\delta < \dot\psi_0 < \frac{\gamma}{1+u_0}. \tag{34}$$

We proceed to show that for $\dot\psi_0$ thus limited, the conditions derived from (33),

$$\begin{aligned} &\dot\psi_0^2 - \frac{3}{2}\delta\dot\psi_0 - \frac{a}{4(1+u_0)} < 0, \\ &\dot\psi_0^2\left(\dot\psi_0 - \frac{3}{2}\delta\right) - \frac{a(a-6\delta^2 u_0)}{12\delta^2(1-u_0^2)}\dot\psi_0 + \frac{a^2}{24\delta(1-u_0^2)} > 0, \end{aligned} \tag{35}$$

are incompatible. If the first of these inequalities be multiplied by $\dot\psi_0$ and subtracted from the second, we find a necessary condition on $\dot\psi_0$ for their consistency which reduces to $\dot\psi_0[3\delta^2(1+u_0) - a] + a\delta/2 > 0$. From (34), recalling that $\gamma = a/2\delta$, we have $a > 4\delta^2(1+u_0)$, so that the coefficient of $\dot\psi_0$ is negative, and

$$\dot\psi_0 < \frac{a\delta/2}{a - 3\delta^2(1+u_0)}.$$

If from this we form the inequality for $\dot\psi_0/2\delta$, and in it substitute $4\delta^2(1+u_0) = a(1-x)$, so that x is always positive, we find as upper bound for $\dot\psi_0/2\delta$ a function of x whose maximum is 1, whereas, by (34), $\dot\psi_0/2\delta$ must be greater than 1. Thus the incompatibility of the conditions (33) is established.

We may state further concerning this case, the consequence of Theorem IX: *the general drift of the longitudinal motion is in the sense which it has at that one of the limiting circles where the curvature is numerically the less.* This is a property whose proof has previously required less elementary methods (see the reference, O, p. 260).

If the spin of the top be stopped again, so that the spherical pendulum is before us, it is known that the difference in longitude between two contacts of the path with its limiting circles exceeds $\pi/2$ and is less than π. The first inequality is obtained by the appraisal of a definite integral. The second has required the use of Cauchy's integral theorem, or other less elementary method (see Appell, loc. cit.). It is readily verified that this second inequality is an immediate consequence of Theorem XI. The hypothesis of the theorem which

interests us may be given the form $\dot{\psi}_0 \varkappa' < 0$, for with the definition of R there employed, the relation (15) is to be replaced by $\varkappa = \operatorname{sgn} \dot{\psi}_0 \cot R$, when we consider a rising arc, $\mathfrak{C}$, of the path. For $\gamma = 0$, (32) takes the form $\varkappa' = -3a^2u'(1-u_0^2)\dot{\psi}_0/4v^5$, so that the hypothesis is fulfilled.

12. **Asymptotic circles.** On page 225 of his article, Professor Osgood mentions an interesting case of motion of the symmetric top, in which it is subjected to a force of constant magnitude, directed always along the positive tangent to the path, $\mathfrak{C}$. He says that $\varkappa$ now approaches 0, and "it is a matter of conjecture as to whether $\mathfrak{C}$ has an asymptotic great circle". Inasmuch as it appears rather obvious that it should have, the statement might seem to be excessively cautious. But the example given in Section 1 shows such caution entirely justified. As a matter of fact, $\mathfrak{C}$ has an asymptotic great circle, by Theorem VII. For, if f denote the magnitude of the force, we find $s = \frac{f}{2A}(t-t_0)^2 + v_0(t-t_0)$, so that the path is infinitely long; also that $\varkappa = Cv/Av$, where $v = \sqrt{v_0^2 + \frac{2f}{A}s}$, so that $\varkappa > 0$ and $\varkappa' < 0$, and hence $K' < 0$. The hypotheses of Theorem VII are therefore fulfilled.

Certain general criteria may be set up for motion with an asymptotic circle. We shall suppose that from some point ($t = t_0$, $s = s_0$) on, v does not vanish, and that the functions involved have whatever continuous derivatives are required. Then the path will be of infinite length if the integral

$$t - t_0 = \int_{s_0}^{s} \frac{ds}{\sqrt{v_0^2 + \frac{2}{A}\int_{s_0}^{s} T\,ds}}$$

is real and finite for all $s > s_0$. This is a first condition.

If we differentiate with respect to s the equation (24_2), and simplify the result by means of (24_1) and (24_3), we obtain

$$Av^2\varkappa' = \frac{dN}{ds} + S - \frac{T(Crv + 2N)}{Av^2}.$$

That the right hand member of this equation, which we assume to be continuous, should never vanish, is the second of the desired conditions.

In the case of a purely tangential force, the latter condition takes the following form: T shall not vanish for $s > s_0$. If T is negative, the motion comes to a halt unless $\int_{s_0}^{s}(-T)\,ds < Av_0^2/2$ for all $s > s_0$. Otherwise, the path has an asymptotic circle, which may, however, reduce to a point provided $T < 0$ and $v \to 0$, i.e. $\int_{s_0}^{\infty}(-T)\,ds = Av_0^2/2$. The asymptotic circle is a great circle if $T > 0$ and $\int_0^{\infty} T ds$ is divergent.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.

THE GREATEST AND THE LEAST VARIATE UNDER GENERAL LAWS OF ERROR*

BY

EDWARD LEWIS DODD

Introduction

To fit frequency distributions, several functions or curves have been used, most of which are generalizations of the so-called normal or Gaussian or Laplacean probability function

$$\frac{h}{\sqrt{\pi}} e^{-h^2x^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}.$$

The differential equation satisfied by this function was generalized by Karl Pearson.† Gram,‡ Charlier,§ and Bruns‖ used the normal function and its successive derivatives, with constant coefficients, to form a series, of which, in practice, only a few terms are used. Jørgensen¶ developed a logarithmic transformation, in which x is replaced by $\log x$. Associated with the Law of Small Numbers is the Poisson exponential function $e^{-\lambda}\lambda^x/x!$ for which Bortkiewicz** gave a four-place table, and Soper†† a six-place table. The Charlier‡‡

* Presented to the Society, April 28, 1923. The word *variate* will refer to any of the particular values which a variable may take on; e. g., the height of some specified soldier in a regiment, — the greatest variate here would be the height of the tallest soldier in the regiment.

† *Contributions to the mathematical theory of evolution,* II: *Skew variation in homogeneous material,* Philosophical Transactions A, vol. 186 (1895), part I, pp. 343-144.

‡ *Über die Entwickelung reeller Funktionen in Reihen mittelst der Methode der kleinsten Quadrate,* Journal für die reine und angewandte Mathematik, vol. 94, pp. 41–73.

§ *Über die Darstellung willkürlicher Funktionen,* Arkiv för Matematik, Astronomi och Fysik, vol. 2, number 20.

‖ *Über die Darstellung von Fehlergesetzen,* Astronomische Nachrichten, vol. 143.

¶ See Arne Fisher, *The Mathematical Theory of Probabilities,* I (2d edition), pp. 236-260.

** *Das Gesetz der Kleinen Zahlen,* 1898. See Arne Fisher, loc. cit., p. 266.

†† Pearson's *Tables for Statisticians and Biometricians,* pp. 113–121.

‡‡ Meddelanden från Lunds Observatorium, 1905. *Vorlesungen über die Grundzüge der Mathematischen Statistik,* p. 6, 79-85. See also Arkiv, loc. cit.

B-Series, for integral variates, makes use of the Poisson function and its differences. The Makeham* life function is well known in life insurance.

Dealing only with the normal function itself, Bortkiewicz† determined mean and modal values for the interval of variation, i. e., the difference between the greatest and the least of n variates. For this same problem there remains to be considered the median and the asymptotic value of the interval of variation. The asymptotic value is a function of n which, with a probability converging to certainty, gives the interval of variation with a relative error small at pleasure. To make the problem broader, we shall consider the greatest and the least variate individually, and shall set up six general classes of functions which include as special cases the frequency functions in common use.

These six classes of functions are distinguished as follows. Apart from a factor $\psi(x)$ satisfying certain inequalities, the probability-function for large values of x is, respectively,

$$(1)\ 0; \quad (2)\ x^{-1-\alpha}; \quad (3)\ g^{x^\alpha}; \quad (4)\ g^{(\log_e x)^\gamma}; \quad (5)\ g^{c^x}; \quad (6)\ x^{-x};$$

with $\alpha > 0, \gamma > 1, 0 < g < 1, c > 1$. The first represents a finite interval; the second is involved in Pearson types; the third, with $\alpha = 2$, is the normal probability function; the fourth leads to logarithmically transformed functions; the fifth, to the Makeham life function; the sixth to the Poisson exponential function.

1. Definitions

Definition 1: Probability function. The function $\varphi(x)$ is a probability function for specified variates, if for each variate

$$\Phi(x) = \int_x^\infty \varphi(t)\,dt$$

is the probability that the variate will take on a value equal to or greater than x.

Even when the statistical material must be given in integers, it is customary to think of $\varphi(x)$ and $\Phi(x)$ as continuous, especially when the number of variates is large. When it is desirable to provide at the same time for

* Journal of the Institute of Actuaries, 1860. See *Institute of Actuaries' Text Book*, Part II, Chap. VI.

† *Variationsbreite und mittlerer Fehler*, Sitzungsberichte der Berliner Mathematischen Gesellschaft, Jahrgang 21, Sitzung am 26. Oktober 1921.

continuous and discontinuous probability, the Stieltjes integral* may be used.

Definition 2: Asymptotic certainty. An "event" dependent upon n variables or variates is asymptotically certain if for any positive η, small at pleasure, it is possible to determine an n' so that when $n > n'$, the probability that the event will happen is greater than $1 - \eta$.

2. The asymptotic value of the greatest variate

THEOREM I. *If the probability function $\varphi(x) \equiv 0$, for $x > x_2$, and if $\int_x^{x_2} \varphi(x)\,dx \neq 0$ when $x < x_2$, then it is asymptotically certain that the greatest of n variates will differ from x_2 by less than any preassigned positive ε.*

Proof. By hypothesis,

$$\int_{x_2-\varepsilon}^{x_2} \varphi(x)\,dx = \delta \neq 0.$$

Then the probability that all the n variates will be less than $x_2 - \varepsilon$ is $(1-\delta)^n$, which approaches zero with increasing n.

A similar statement can be made for the least variate. In fact, in all the theorems which follow the treatment of the least variate will be omitted as obvious. Of course, x will often be replaced here by $|x|$.

THEOREM II. *If, for positive x, the probability function is*†

$$\varphi(x) = x^{-1-\alpha} \cdot \psi(x),$$

with α, k_1, k_2, positive constants, and $k_1 < \psi(x) < k_2$, then it is asymptotically certain that the greatest of n variates will be

$$n^{(1+\varepsilon')/\alpha},$$

where $|\varepsilon'| < \varepsilon$, small at pleasure.

---

* See R. von Mises, *Fundamentalsätze der Wahrscheinlichkeitsrechnung*, Mathematische Zeitschrift, vol. 4 (1919), pp. 1-97.

† Here, and in the following theorems, if $\varphi(x)$ has the indicated form merely when x is greater than some given constant, the theorem remains valid.

Proof. By hypothesis,

$$\int_x^\infty \varphi(x)\,dx < k_2 \int_x^\infty x^{-1-\alpha}\,dx = \frac{k_2}{\alpha} x^{-\alpha}.$$

Hence, if with $\varepsilon > 0$ small at pleasure, we set

$$x = n^{(1+\varepsilon)/\alpha}$$

it follows that

$$\int_x^\infty \varphi(x)\,dx < \frac{k_2}{\alpha n^{1+\varepsilon}}.$$

Thus, the probability that a specified variate will be less than this x is greater than

$$1 - \frac{k_2}{\alpha n^{1+\varepsilon}}.$$

And the probability that all variates will be less than this x is greater than

$$\left(1 - \frac{k_2}{\alpha n^{1+\varepsilon}}\right)^n.$$

But this approaches unity as n approaches infinity. And thus, with $\eta > 0$ small at pleasure, it is possible to find n' so that if $n > n'$, the probability that all variates will be less than $n^{(1+\varepsilon)/\alpha}$ is greater than $1 - \frac{1}{2}\eta$.

Similarly, using

$$\lim_{n\to\infty} \left(1 - \frac{k_1}{\alpha n^{1-\varepsilon}}\right)^n = 0,$$

it can be shown that the probability that all variates will be less than

$$n^{(1-\varepsilon)/\alpha}$$

is less than $\frac{1}{2}\eta$ for n greater than some n''.

Thus, for large enough n, the greatest variate will lie in the interval from $n^{(1-\varepsilon)/\alpha}$ to $n^{(1+\varepsilon)/\alpha}$, unless all variates are less than $n^{(1-\varepsilon)/\alpha}$, — for which the probability is less than $\frac{1}{2}\eta$, — or unless some variate surpasses $n^{(1+\varepsilon)/\alpha}$, — for which the probability is likewise less than $\frac{1}{2}\eta$. Hence, by Definition 2, it is asymptotically certain that the greatest variate will lie in the interval from $n^{(1-\varepsilon)/\alpha}$ to $n^{(1+\varepsilon)/\alpha}$.

THEOREM III. *If the probability function is*

$$\varphi(x) = g^{x^\alpha}\cdot\psi(x), \quad \textit{with} \quad x^{-\beta} < \psi(x) < x^\beta,$$

where α, β, g are positive constants, and $g<1$, then it is asymptotically certain that the greatest of n variates will equal

$$(-\log_g n)^{1/\alpha}(1+\varepsilon'), \quad \textit{with} \quad |\varepsilon'| < \varepsilon,$$

small at pleasure.

Proof. Let

$$I = \int_x^\infty g^{t^\alpha} t^\beta\, dt, \qquad v = g^{t^\alpha}, \qquad u = \frac{t^{\beta-\alpha+1}}{\alpha \log_e g}.$$

Then, integrating by parts,

$$I = Cg^{x^\alpha}x^{\beta-\alpha+1} + C(\beta-\alpha+1)\int_x^\infty g^{t^\alpha}\, t^{\beta-\alpha}\, dt; \qquad C = \frac{1}{\alpha(-\log_e g)} > 0.$$

Thus,

$$I < Cg^{x^\alpha}\, x^{\beta-\alpha+1}$$

provided $\beta-\alpha+1<0$. But, even if $\beta-\alpha+1>0$, the process performed k times will put into the numerator $\beta-k\alpha+1$, which is ultimately negative. Hence

$$I < g^{x^\alpha}\cdot F(x),$$

where $F(x)$ is a sum of powers of x with constant coefficients.

Suppose, now, that

$$x = (-\log_g n)^{1/\alpha}(1+\varepsilon),$$

$\varepsilon > 0$, small at pleasure. Then,

$$g^{x^\alpha} = \frac{1}{n^{1+2\varepsilon_1}}$$

where $1+2\varepsilon_1 = (1+\varepsilon)^\alpha$, $\varepsilon_1 > 0$. But, for large enough n, $F(x) < n^{\varepsilon_1}$. Then

$$I < \frac{1}{n^{1+\varepsilon_1}}.$$

Hence, the probability that the n variates will all be less than x is greater than

$$\left(1-\frac{1}{n^{1+\varepsilon_1}}\right)^n > 1-\tfrac{1}{2}\eta$$

for $n >$ some n'.

Now, let J be the result of replacing β by $-\beta$ in I. If, after integrating by parts, $-\beta-\alpha+1>0$, then

$$J > C g^{x^\alpha} x^{-\beta-\alpha+1}.$$

But if $-\beta-\alpha+1<0$, a second integration by parts yields $-\beta-2\alpha+1$, which is again negative. By combining the two results,

$$J > G(x)\cdot g^{x^\alpha},$$

where $G(x)$ is a sum of powers of x with constant coefficients. If, now,

$$x = (-\log_g n)^{1/\alpha}(1-\varepsilon),$$

then for large n,

$$J > 1-\frac{G(x)}{n^{1-2\varepsilon_2}} > 1-\frac{1}{n^{1-\varepsilon_2}},$$

where $1-2\varepsilon_2 = (1-\varepsilon)^a$, $\varepsilon_2 > 0$. Thus, the probability that the n variates will all be less than x is less than

$$\left(1-\frac{1}{n^{1-\varepsilon_2}}\right)^n < \tfrac{1}{2}\eta$$

when $n >$ some n''.

THEOREM IV. *If the probability function is*

$$\varphi(x) = g^{(\log_c x)^\gamma} \cdot \psi(x),$$

with $x^{-\beta} < \psi(x) < x^\beta$, *where* c, g, β, γ *are positive constants,* $0 < g < 1$, $c > 1$, $\gamma > 1$, *then it is asymptotically certain that the greatest of* n *variates will equal*

$$c^{(-\log_g n)^{(1+\varepsilon')/\gamma}},$$

with $|\varepsilon'| < \varepsilon$, *small at pleasure.*

Proof. The proof follows the same general course as in the preceding theorem, with

$$v = g^{(\log_c t)^\gamma}, \qquad I = \int_x^\infty v\, t^\beta\, dt, \qquad J = \int_x^\infty v\, t^{-\beta}\, dt.$$

Upon integrating I by parts, the new integral contains the same integrand multiplied by a factor which is increased if the negative portion is dropped, and t is replaced by x. Thus

$$I < \xi(x) + \zeta(x)\cdot I,$$

where, indeed, $\zeta(x) < 1$ when x is large and where the principal factors of $\xi(x)$ are $g^{(\log_c x)^\gamma}$ and powers of x. As for J, we may first take $\beta > 1$, and again in the new factor set $t = x$.

Now, setting

$$g^{(\log_c x)^\gamma} = \frac{1}{n^m}, \qquad g^{(\log_c x)^\gamma}\cdot x^\tau = \frac{1}{n^{m'}},$$

it follows that

$$(\log_c x)^\gamma = -m \log_g n, \qquad (\log_c x)^\gamma + \tau \log_g x = -m' \log_g n.$$

By division,

$$1+\frac{\tau \log_c x\cdot \log_g c}{(\log_c x)^\gamma}=\frac{m'}{m}.$$

Hence, since $\gamma>1$,

$$\lim_{x\to\infty}\frac{m'}{m}=1.$$

Thus, asymptotically, the effect of x^τ disappears when combined with $g^{(\log_c x)^\gamma}$.

Now set

$$x=c^{(-\log_g n)^{(1+\varepsilon)/\gamma}}.$$

Then

$$(\log_c x)^\gamma=(-\log_g n)^{1+\varepsilon}>(1+\varepsilon)(-\log_g n),$$

provided n is large enough. Thus, since $g<1$,

$$g^{(\log_c x)^\gamma}<\frac{1}{n^{1+\varepsilon}}.$$

With ε_1 suitably chosen, positive and less than ε, the probability that all variates will be less than x is greater than

$$\left(1-\frac{1}{n^{1+\varepsilon_1}}\right)^n,$$

which approaches unity as a limit.

THEOREM V. *If the probability function is*

$$\varphi(x)=g^{c^x}\cdot\psi(x),\ \textit{with}\ b^{-x}<\psi(x)<b^x,$$

where b, c, and g are constants, $0<g<1$, $b>1$, $c>1$, then it is asymptotically certain that the greatest of n variates will equal

$$[\log_c(-\log_g n)](1+\varepsilon'),\ \textit{with}\ |\varepsilon'|<\varepsilon,$$

small at pleasure.

Proof. The proof follows the same general course as in Theorem III, with

$$v = g^{c^t}, \quad I = \int_x^\infty v\, b^t\, dt, \quad J = \int_x^\infty v\, b^{-t}\, dt, \quad b_1 = \frac{b}{c}.$$

Then, upon integrating by parts, we find

$$I < \frac{g^{c^x}\, b_1^x}{(-\log g)(\log c)} \quad \text{if } \frac{b}{c} < 1.$$

At all events, $\frac{b}{c^k} < 1$, for large enough k, so that by repetitions of the process,

$$I < g^{c^x} \cdot F(x),$$

where $F(x)$ is a sum of terms such as b_1^x with constant coefficients. Likewise J can be proved greater than a similar expression, noting that $\log(bc) > 0$. By first setting $x = [\log_c(-\log_g n)](1+\varepsilon)$, and then $x = [\log_c(-\log_g n)](1-\varepsilon)$ and noting that for large n, $(-\log_g n) < n^\delta$, $\delta > 0$, small at pleasure, the required inequalities can be obtained.

THEOREM VI. *If the probability function is*

$$\varphi(x) = x^{-x} \cdot \psi(x), \text{ with } b^{-x} < \psi(x) < b^x, \text{ constant } b > 1,$$

then it is asymptotically certain that the greatest of n variates will be

$$X(1+\varepsilon'), \text{ where } X^X = n, \text{ with } |\varepsilon'| < \varepsilon,$$

small at pleasure.

Proof. Set

$$v = t^{-t}, \qquad I = \int_x^\infty v\, b^t\, dt, \qquad J = \int_x^\infty v\, b^{-t}\, dt.$$

Integrating by parts, dropping a negative term, replacing t by x, we obtain

$$I < z x^{-x} b^x + I z \log b, \text{ where } z = \frac{1}{1 + \log x}.$$

Likewise,

$$J > zx^{-x}b^{-x} - \{z\log b + z^2 x^{-1}\}\,J.$$

Moreover, if $x^x = n^y$ and $x^x b^x = n^{y'}$, then asymptotically $\dfrac{dx}{x} = \dfrac{dy}{y}$, $y' = y$. That is, a percentage error in y is controlled by an equal percentage error in x; and b^x, when combined with x^x, makes no asymptotic contribution to the exponent of n.

3. The median* value of the greatest variate

The probability that every one of n variates will be less than G is, as is well known,

$$\left(1 - \int_G^\infty \varphi(t)\,dt\right)^n,$$

where $\varphi(x)$ is the probability function. If, now, we determine G so that this expression is equal to $\frac{1}{2}$, then it is equally likely that the greatest variate will or will not exceed G. This median value of the greatest variate is thus obtained by finding G so that

$$\int_G^\infty \varphi(t)\,dt = 1 - 2^{-1/n}.$$

While a median, mean, or modal value for the greatest variate may be more difficult to compute than the asymptotic value, in the foregoing theorems, the former will, in general have more significance in practical problems. However, even here, the asymptotic value may be useful for a rough simple check.

4. The normal† probability function

If, in Theorem III, we set

$$h^2 = \frac{1}{2\sigma^2} = -\log_e g, \qquad \alpha = 2,$$

* In the theory of errors, the so-called "probable error" is the median of the absolute values of the errors. Thus, it is equally likely that an error, taken positively, will or will not exceed the probable error.

† Rietz, in his article *Frequency distributions obtained by certain transformations of normally distributed variates,* Annals of Mathematics, ser. 2, vol. 23 (1922), pp. 292–300,

then

$$g^{x^{\alpha}} = e^{-h^2x^2} = e^{-x^2/2\sigma^2}.$$

Hence, under the normal probability law, it is asymptotically certain that the greatest variate will be, apart from the factor $(1 + \epsilon')$,

$$(-\log_g n)^{1/\alpha} = \frac{\sqrt{\log_e n}}{h} = \sigma\sqrt{2\log_e n}.$$

These results hold also for a Gram series with but a finite number of terms, since the polynomial factor has no asymptotic influence.

On account of the symmetry of the normal function, an average value for the variation interval is obtained by doubling the corresponding value for the greatest variate. The following table compares the median and asymptotic values of the variation interval, computed by the formulas of this paper, with the modal, mean, and restricted mean ("bedingte . . . mathematische Erwartung") values obtained by Bortkiewicz.* Bortkiewicz, indeed, after noting that his mean determination is very close to the average of the other two, gives examples from anthropometry and roulette in which the actual variation is close to his mean value.

Values of $\frac{\text{Variation Interval}}{\sigma}$ *for n variates subject to* $\frac{1}{\sigma\sqrt{2\pi}}\,e^{-x^2/2\sigma^2}$

Number of variates n	Modal value (Bortkiewicz)	Median value (Dodd)	Mean value (Bortkiewicz)	Restricted mean value (Bortkiewicz)	Asymptotic (Dodd)	Asymptotic ÷ Median
100	4.76	4.92	5.04	5.30	6.07	1.23
1,000	6.23	6.39	6.48	6.73	7.43	1.16
10,000	7.47	7.62	7.70	7.92	8.58	1.13
100,000	8.55	8.69	8.70	8.95	9.60	1.10

The asymptotic value of the interval of variation is thus about 23% too large when $n = 100$, and is still 10% too large when $n = 100{,}000$.

considers in particular the transformation $x'' = kx^n$, which would, for example, give the distribution of volumes of similar solids — "oranges" — if the "diameters" are normally distributed. In such a case as this, where x'' is an increasing function of x, the asymptotic value of the greatest "volume" can be found by finding first the asymptotic value of the greatest "diameter".

* Loc. cit. See also Nordisk Statistisk Tidskrift, vol. 1, pp. 11-38.

5. THE PEARSON FREQUENCY TYPES*

The Pearson frequency types are given below with the asymptotic interval of variation for each, and the number of the theorem involved.

Asymptotic interval of variation for the Pearson frequency types

Type number	Frequency function	Interval of variation	Number of theorem
I	$y_0\left(1+\frac{x}{\alpha}\right)^{\nu\alpha}\left(1-\frac{x}{\beta}\right)^{\nu\beta}$	$\alpha+\beta$	I
II	$y_0\left(1-\frac{x^2}{\alpha^2}\right)^{\nu\alpha}$	2α	I
III	$y_0\left(1+\frac{x}{\alpha}\right)^{\nu\alpha}e^{-\nu x}$	$\alpha+\frac{\log_e n}{\nu}$	I, III
IV	$y_0\left(1+\frac{x^2}{a^2}\right)^{-m}e^{-\nu\tan^{-1}(x/a)}$	$2\,n^{1/(2m-1)},\ m>\frac{1}{2}$	II
V	$y_0\,x^{-s}\,e^{-\gamma/x},\ x>0$	$n^{1/(s-1)},\ s>1$	I, II
VI	$y_0\,(x-a)^{q_2}x^{-q_1},\ x\geqq a$	$n^{1/(q_1-q_2-1)}-a,\ q_1>q_2+1$	I, II
VII	$y_0\,e^{-x^2/2\sigma^2}$	$2\sigma\sqrt{2\log_e n}$	III

6. OTHER FREQUENCY FUNCTIONS

1. **Jørgensen function.** The Jørgensen function is of the form†

$$ke^{-\frac{1}{2}\left[\frac{\log x-\xi}{\delta}\right]^2},$$

where k, ξ, and δ are constants, and $x>0$. It can be written

$$k'x^{\beta'}g^{(\log x)^2},$$

where

$$\log_e g=\frac{-1}{2\delta^2},\qquad \beta'=\frac{\xi}{\delta^2},\qquad k'=\text{constant}.$$

* Certain special and limiting cases have also been designated as "types".

† See Arne Fisher, loc. cit., p. 241.

Then, by Theorem IV, the asymptotic value of the greatest variate is

$$e^{\delta\sqrt{2\log_e n}}(1+\varepsilon'), \text{ with } |\varepsilon'|<\varepsilon,$$

small at pleasure.

2. **Poisson exponential function and Charlier *B*-curve.** The Poisson function has the form

$$\frac{e^{-\lambda}\lambda^x}{x!}.$$

But, by Stirling's formula,

$$x! = x^x e^{-x}\sqrt{2\pi x}\, e^{\theta/12x}, \qquad 0<\theta<1.$$

For the asymptotic value of the greatest variate, the only significant factor here is x^x. This asymptotic value, by Theorem VI, is

$$X(1+\varepsilon'), \text{ where } X^X = n, \ |\varepsilon'|<\varepsilon,$$

small at pleasure. The Charlier *B*-curve for integral variates is obtained from the Poisson function by differencing. But

$$x^x-(x-1)^{x-1} = x^x\left[1-\frac{\left(1-\frac{1}{x}\right)^x}{x-1}\right],$$

and this bracket has no asymptotic significance. Hence, if only a finite number of terms are taken, the greatest variate, asymptotically, remains that determined as above.

3. **Makeham life function.** The Makeham formula for the number of survivors at age x from an original group of l_0 individuals just born is

$$l_x = k s^x g^{c^x},$$

where $k>0$, $0<s<1$, $0<g<1$, $c>1$. Postulating a stable population supported by the same number of births annually, and assuming that the theoretic relative frequency is the equivalent of probability, the following table,

based upon Theorem V, gives the age of the oldest individual, in accordance with constants used in the American Experience Table, as makehamized by Arthur Hunter,* in the Institute of Actuaries Table (H^M), and in the McClintock Annuitant Tables, makehamized by W. M. Strong.*

Age of oldest of n individuals

By asymptotic formula $\log_c(-\log_g n)$, where $l_x = ks^x g^{c^x}$.

Makehamized mortality table	For population of		
	one thousand	one million	one billion
	oldest age	oldest age	oldest age
American experience	95.1	101.7	105.5
Institute of Actuaries	96.3	103.9	108.3
McClintock-Male	97.8	105.3	109.7
McClintock-Female	100.8	108.3	112.7

While these results are somewhat crude, it seems surprising that the asymptotic formula which dispenses with the factor s^x could do so well. The question, indeed, arises whether any graduation formula can throw much light upon extreme ages, because of the gross irregularities commonly found at the ends of biologic series.

7. Summary

The interval of variation is the difference between the greatest and the least of n variates in a distribution. Theorems are here given for the greatest variate; corresponding theorems can be stated for the least variate, using $|x|$ in place of x when necessary. In the following table which summarizes these theorems, the letters stand for positive numbers; they are constants except x, n, and G. Moreover, $g < 1$; but $b > 1$, $c > 1$, $\gamma > 1$. For each variate $\int_x^\infty \varphi(x)\,dx$ is the probability that the variate will be equal to or greater than x. With $\varphi(x) = \varphi_1(x)\cdot\psi(x)$, the two factors are each described below. As n increases indefinitely, a probability converges to certainty that the greatest variate will take on the stated asymptotic value, with a relative error small at pleasure for the values in Classes I, III, V, and VI, and for $1/\alpha$ and $1/\gamma$ in Classes II and IV.

* Transactions of the Actuarial Society of America, vol. 7, p. 200, p. 289.

Asymptotic value of the greatest of n variates

When each variate is subject to $\varphi \cdot (x) \cdot \psi(x)$.*

Class	$\varphi_1(x)$	Conditions for $\psi(x)$	Greatest variate†	Applications
I	$0,\ x > x_2$	—	x_2‡	5 Pearson Types
II	x^{-1-a}	$k_1 < \psi < k_2$	$n^{1/a}$	3 Pearson Types
III	g^{x^a}	$x^{-\beta} < \psi < x^{\beta}$	$(-\log_g n)^{1/a}$	Gaussian Function Grams Series (finite)
IV	$g^{(\log_c x)^\gamma}$	$x^{-\beta} < \psi < x^{\beta}$	$c^{(-\log_g n)^{1/\gamma}}$	Jørgensen Logarithmic Function
V	g^{c^x}	$b^{-x} < \psi < b^{x}$	$\log_c(-\log_g n)$	Makeham Life Function
VI	x^{-x}	$b^{-x} < \psi < b^{x}$	G, with $G^G = n$	Poisson Exponential Charlier *B*-Series (finite)

Asymptotic values have a theoretic importance because of the rigidity of the determination. Possibly, they may be used unreservedly in problems where the variates are as numerous as atoms; but in most practical problems, their chief value would seem to be in furnishing a rough check upon mean, modal, or median values. The latter can be found by determining G so that

$$\int_G^\infty \varphi(x)\,dx = 1-2^{-1/n}.$$

* Or merely subject when x is sufficiently large.

† Apart from the factor $(1+\varepsilon')$, with $|\varepsilon'| < \varepsilon$, small at pleasure.

‡ Provided $\int_{x_2-\varepsilon}^{x_2} \varphi_1(x) \neq 0$.

UNIVERSITY OF TEXAS,
AUSTIN, TEX.

based upon Theorem V, gives the age of the oldest individual, in accordance with constants used in the American Experience Table, as makehamized by Arthur Hunter,* in the Institute of Actuaries Table (H^M), and in the McClintock Annuitant Tables, makehamized by W. M. Strong.*

Age of oldest of n individuals

By asymptotic formula $\log_c(-\log_g n)$, where $l_x = ks^x g^{c^x}$.

Makehamized mortality table	For population of one thousand	one million	one billion
	oldest age	oldest age	oldest age
American experience	95.1	101.7	105.5
Institute of Actuaries	96.3	103.9	108.3
McClintock-Male	97.8	105.3	109.7
McClintock-Female..................	100.8	108.3	112.7

While these results are somewhat crude, it seems surprising that the asymptotic formula which dispenses with the factor s^x could do so well. The question, indeed, arises whether any graduation formula can throw much light upon extreme ages, because of the gross irregularities commonly found at the ends of biologic series.

7. Summary

The interval of variation is the difference between the greatest and the least of n variates in a distribution. Theorems are here given for the greatest variate; corresponding theorems can be stated for the least variate, using $|x|$ in place of x when necessary. In the following table which summarizes these theorems, the letters stand for positive numbers; they are constants except x, n, and G. Moreover, $g < 1$; but $b > 1$, $c > 1$, $\gamma > 1$. For each variate $\int_x^\infty \varphi(x)\,dx$ is the probability that the variate will be equal to or greater than x. With $\varphi(x) = \varphi_1(x)\cdot\psi(x)$, the two factors are each described below. As n increases indefinitely, a probability converges to certainty that the greatest variate will take on the stated asymptotic value, with a relative error small at pleasure for the values in Classes I, III, V, and VI, and for $1/\alpha$ and $1/\gamma$ in Classes II and IV.

* Transactions of the Actuarial Society of America, vol. 7, p. 200, p. 289.

Asymptotic value of the greatest of n variates

When each variate is subject to $\varphi \cdot (x) \cdot \psi(x)$.*

Class	$\varphi_1(x)$	Conditions for $\psi(x)$	Greatest variate†	Applications
I	$0,\ x > x_2$	—	x_2‡	5 Pearson Types
II	x^{-1-a}	$k_1 < \psi < k_2$	$n^{1/a}$	3 Pearson Types
III	g^{x^a}	$x^{-\beta} < \psi < x^\beta$	$(-\log_g n)^{1/a}$	Gaussian Function Grams Series (finite)
IV	$g^{(\log_c x)^\gamma}$	$x^{-\beta} < \psi < x^\beta$	$c^{(-\log_g n)^{1/\gamma}}$	Jørgensen Logarithmic Function
V	g^{c^x}	$b^{-x} < \psi < b^x$	$\log_c(-\log_g n)$	Makeham Life Function
VI	x^{-x}	$b^{-x} < \psi < b^x$	G, with $G^G = n$	Poisson Exponential Charlier B-Series (finite)

Asymptotic values have a theoretic importance because of the rigidity of the determination. Possibly, they may be used unreservedly in problems where the variates are as numerous as atoms; but in most practical problems, their chief value would seem to be in furnishing a rough check upon mean, modal, or median values. The latter can be found by determining G so that

$$\int_G^\infty \varphi(x)\,dx = 1 - 2^{-1/n}.$$

* Or merely subject when x is sufficiently large.

† Apart from the factor $(1+\varepsilon')$, with $|\varepsilon'| < \varepsilon$, small at pleasure.

‡ Provided $\int_{x_2-\varepsilon}^{x_2} \varphi_1(x) \neq 0$.

UNIVERSITY OF TEXAS,
AUSTIN, TEX.

THE INTERSECTION NUMBERS*

BY

OSWALD VEBLEN

1. In his first memoir on analysis situs† Poincaré defined a number $N(\Gamma_k, \Gamma_{n-k})$ which had previously been considered, at least in special cases, by Kronecker. With certain conventions as to sign this number represents the excess of the number of positive over the number of negative intersections of a k-dimensional circuit Γ_k with an $(n-k)$-dimensional circuit Γ_{n-k} when both are immersed in an n-dimensional oriented manifold. The purpose of the present paper is to show how to calculate this number when the manifold is defined combinatorially as a collection of cells and the circuits are composed of sets of these cells; and to show how the matrices which represent the intersectional relations between the k-circuits and the $(n-k)$-circuits depend on the matrices of orientation of the manifold. We also define certain modulo 2 intersection numbers and discuss the matrices connected with them.

The terminology and notations of the *Cambridge Colloquium Lectures on Analysis Situs* (New York, 1922) will be used without further explanation, and the references not otherwise indicated will be to that book.

2. Let a manifold M_n be given as the set of all points of a complex C_n. Let C'_n be a complex dual to C_n constructed as explained on page 88 by means of a complex $\overline{C}_n$ which is a regular subdivision both of C_n and of C'_n. Every k-cell a^k_j of C_n has a single point P^k_j (cf. p. 85) in common with a single $(n-k)$-cell of C'_n which is called b^{n-k}_j. Our first problem will be to assign a positive or negative sign to the intersection of a^k_j with b^{n-k}_j.

In order to do this, we suppose M_n to be oriented as explained in Chapter IV and that all cells, circuits, etc., are oriented. Moreover, in the regular complex $\overline{C}_n$, in which each i-cell is uniquely determined by its $i+1$ vertices, the orientation of the i-cell will be denoted by the order in which its vertices are written, and the following two conventions will be followed: (1) if $A_0 A_1 \cdots A_k$ denotes a given oriented k-cell $(k = 1, 2, \ldots, n)$ any even permutation of $A_0 A_1 \cdots A_k$ denotes the same oriented k-cell and any odd permutation denotes its negative; (2) the oriented $(k-1)$-cell $A_1 A_2 \cdots A_k$ is positively related to the oriented k-cell $A_0 A_1 \cdots A_k$.

---

* Presented to the Society under a different title, April 24, 1920.

† Journal de l'École Polytechnique, ser. 2, vol. 1 (1895).

A simple argument by mathematical induction could, but will not here, be given to prove that these notations and conventions are consistent with themselves and with the definition of oriented cells.

3. The k-cell a_j^k of C_n is made up of a number of k-cells of $\overline{C}_n$ having P_j^k as their common vertex. Using the notation of page 86, let one of these be denoted by

$$P_a^0\, P_b^1 \cdots P_i^{k-1}\, P_j^k,$$

the points P being chosen, as is always possible, so that the orientation of this k-cell agrees with that of a_j^k. In like manner, b_j^{n-k} is made up of a number of $(n-k)$-cells of $\overline{C}_n$ having P_j^k as their common vertex, and we let any one of these be denoted by

$$P_j^k\, P_l^{k+1} \cdots P_s^n,$$

the points P being chosen this time so that the sense of the k-cell which they represent agrees with that of b_j^{n-k}. According as the oriented n-cell

$$P_a^0\, P_b^1 \cdots P_i^{k-1}\, P_j^k\, P_l^{k+1} \cdots P_s^n$$

is positively or negatively oriented, we say that the intersection of a_j^k with b_j^{n-k} is positive or negative. In the first case we write

$$N(a_j^k, b_j^{n-k}) = 1$$

and in the second case

$$N(a_j^k, b_j^{n-k}) = -1.$$

From the definition of the points P it follows directly that this definition is independent of the particular cells of $\overline{C}_n$ which it employs. It also follows that the function N is such that

$$\begin{aligned} N(a_j^k, b_j^{n-k}) &= -N(-a_j^k, b_j^{n-k}) \\ &= -N(a_j^k, -b_j^{n-k}). \end{aligned} \tag{3.1}$$

Since the relation between C_n and C_n' is reciprocal, the definition given here determines the meaning of $N(b_j^{n-k}, a_j^k)$, and a simple count of transpositions in the notation gives the formula

$$N(b_j^{n-k}, a_j^k) = (-1)^{k(n-k)}\, N(a_j^k, b_j^{n-k}). \tag{3.2}$$

4. The cells of C_n and $\bar{C}_n$ are so oriented (cf. p. 123) that

$$E_k' = \bar{E}_{n-k+1},$$

which means that a_j^k is positively or negatively related to a_i^{k-1} according as b_i^{n-k+1} is positively or negatively related to b_j^{n-k}. Now the points P may be so chosen that $P_a^0 P_b^1 \cdots P_i^{k-1}$ represents an oriented cell on a_i^{k-1} and $P_i^{k-1} P_j^k P_l^{k+1} \cdots P_s^n$ represents an oriented cell on b_i^{n-k+1}. By the definition in § 2 above, the oriented cell $P_a^0 P_b^1 \cdots P_i^{k-1}$ is positively or negatively related to $P_a^0 P_b^1 \cdots P_i^{k-1} P_j^k$, and therefore to a_i^k, according as $(-1)^k$ is positive or negative. On the other hand, $P_i^{k-1} P_j^k P_l^{k+1} \cdots P_s^n$ is positively related to $P_j^k P_l^{k+1} \cdots P_s^n$, and therefore to b_i^{n-k}. Hence if b_i^{n-k+1} is positively related to b_i^{n-k}, a_i^{k-1} is positively related to a_i^k and $(-1)^k N(a_i^{k-1}, b_i^{n-k+1})$ is positive or negative according as

$$P_a^0 P_b^1 \cdots P_i^{k-1} P_j^k P_l^{k+1} \cdots P_s^n$$

is positively or negatively oriented. A similar result holds if b_i^{n-k+1} is negatively related to b_j^{n-k}. Hence

$$N(a_j^k, b_j^{n-k}) = (-1)^k N(a_i^{k-1}, b_i^{n-k+1}).$$

By repeated application of this formula we obtain

$$N(a_j^k, b_j^{n-k}) = (-1)^{k(k+1)/2} N(a_a^0, b_a^n).$$

But all the n-cells b_i^n are similarly oriented. Hence the value of $N(a_a^0, b_a^n)$ is the same for all zero cells a_a^0, and consequently the value of $N(a_j^k, b_j^{n-k})$ is independent of j. Hence if the notation is so chosen that b_i^n is positively oriented,*

$$\begin{aligned} N(a_i^0, b_i^n) &= 1, \\ N(a_i^1, b_i^{n-1}) &= -1, \\ N(a_i^2, b_i^{n-2}) &= 1, \\ N(a_i^3, b_i^{n-3}) &= -1, \\ &\vdots \end{aligned}$$

and all these equations are independent of i.

* Cf. Poincaré, Proceedings of the London Mathematical Society, vol. 32 (1900) p. 280.

5. An oriented complex Γ_k composed of the oriented k-cells $a_1^k\, a_2^k \cdots a_{\alpha_k}^k$ counted x^1 times, x^2 times, $\cdots$, x^{α_k} times, respectively, is represented by the notation

$$\Gamma_k = (x^1, x^2, \ldots, x^{\alpha_k}). \tag{5.1}$$

Let Γ'_{n-k} be an arbitrary oriented complex of C'_k, so that

$$\Gamma'_{n-k} = (y^1, y^2, \ldots, y^{\alpha_k}). \tag{5.2}$$

By the number of intersections of Γ_k with Γ'_{n-k}, having regard to sign, we shall mean the number $N(\Gamma_k, \Gamma'_{n-k})$ defined by means of the equation

$$\begin{aligned} N(\Gamma_k, \Gamma'_{n-k}) &= \sum_{j=1}^{\alpha_k} x^j\, y^j\, N(a_j^k\, b_j^{n-k}) \\ &= (-1)^{k(k+1)/2} \sum_{j=1}^{\alpha_k} x^j\, y^j. \end{aligned} \tag{5.3}$$

If we recall that there are no intersections of cells of Γ_k of dimensionality less than k with cells of Γ'_{n-k} and that no cell a_i^k intersects a cell b_j^{n-k} unless $i = j$, it is clear that this definition is in accordance with geometric intuition.

6. The last equation has as obvious corollaries the equations

$$N(\Gamma_k + \Delta_k, \Gamma'_{n-k}) = N(\Gamma_k, \Gamma'_{n-k}) + N(\Delta_k, \Gamma'_{n-k}), \tag{6.1}$$

$$N(\Gamma_k, \Gamma'_{n-k} + \Delta'_{n-k}) = N(\Gamma_k, \Gamma'_{n-k}) + N(\Gamma_k, \Delta'_{n-k}), \tag{6.2}$$

from which it follows that if $\Gamma_k^i\,(i = 1, 2, \ldots, \alpha_k)$ is any set of k-dimensional complexes on which all k-dimensional complexes of C_n are linearly dependent and $\Gamma_{n-k}^i\,(i = 1, 2, \ldots, \alpha_k)$ a set of $(n-k)$-dimensional complexes on which all $(n-k)$-dimensional complexes of C'_n are linearly dependent, then if

$$\Gamma_k = \sum_{i=1}^{\alpha_k} x_i\, \Gamma_{n-k}^i \tag{6.3}$$

and

$$\Gamma_{n-k} = \sum_{i=1}^{\alpha_k} y_j\, \Gamma_{n-k}^j \tag{6.4}$$

where the x's and y's are integers, then

$$N(\Gamma_k, \Gamma_{n-k}) = \sum_{i=1}^{a_k}\sum_{j=1}^{a_k} x_i\, y_j\, N(\Gamma_k^i, \Gamma_{n-k}^j). \tag{6.5}$$

Hence the intersection numbers of all k-dimensional complexes with all $(n-k)$-dimensional complexes depend on the matrix of numbers $N(\Gamma_k^i, \Gamma_{n-k}^j)$. By choosing the complexes Γ_k^i and Γ_{n-k}^j in the normal manner described in the Colloquium Lectures this matrix may be given a very simple form, which we shall determine in the next three sections.

7. As proved on page 116 of the Colloquium Lectures, a set of k-dimensional complexes upon which all the complexes formed from cells of C_n are linearly dependent may be so chosen as to consist of (1) a set of P_{k-1} non-bounding circuits which we shall denote by $\Gamma_k^i (i = 1, \cdots, P_k - 1)$, or in Poincaré's notation,

$$\Gamma_k^i \equiv 0; \tag{7.1}$$

(2) a set of τ_k circuits Λ_k^i $(i = 1, \ldots, \tau_k)$ which satisfy the homologies

$$t_i^k \Lambda_k^i \sim 0 \tag{7.2}$$

in which t_i^k represents a k-dimensional coefficient of torsion; (3) a set of $r_{k+1} - \tau_k$ bounding circuits Θ_k^i

$$\Theta_k^i \sim 0; \tag{7.3}$$

and (4) and (5) two sets of complexes Φ_k^i and Ψ_k^i which are not circuits but satisfy the following congruences:

$$\Phi_k^i \equiv \Theta_{k-1}^i, \qquad 0 < i < r_k - \tau_{k-1}, \tag{7.4}$$

$$\Psi_k^i \equiv t_i^{k-1} \Lambda_{k-1}^i, \qquad 0 < i < \tau_{k-1}, \tag{7.5}$$

in which Θ_{k-1}^i and Λ_{k-1}^i are defined by replacing k by $k-1$ in (7.3) and (7.2).

These relations are derived from the matrix equation

$$E_k \cdot D_k = C_{k-1} \cdot E_k^* \tag{7.6}$$

which arises in reducing (cf. p. 108) the orientation matrix E_k to normal form. The matrix E_k^* is one in which all elements are zero except the first r_k elements of the main diagonal. The first $r_k - \tau_{k-1}$ of the non-zero elements are 1 and the remaining τ_{k-1} are the coefficients of torsion of dimensionality $k-1$.

The first $r_k - \tau_{k-1}$ columns of D_k represent the complexes Φ_k^i, the next τ_{k-1} columns represent the complexes Ψ_k^i, the next $P_k - 1$ columns represent the circuits Γ_k^i, the next $r_{k+1} - \tau_k$ columns represent the circuits Θ_k^i, the next τ_k columns represent the circuits Λ_k^i. Thus, for example, if the jth column of D_k $(0 < j < r_k - \tau_{k-1})$ is $(x_{1j}, x_{2j}, \ldots, x_{\alpha_k j})$ we have

$$\Phi_k^j = (x_{1j}, x_{2j}, \ldots, x_{\alpha_k j}). \tag{7.7}$$

The columns of the matrix C_{k-1} are the same as the columns of D_{k-1} in a different order, and each complex represented by a column of D_k is bounded by the circuit represented by the corresponding column of the matrix $C_{k-1} \cdot E_k^*$. It is from this fact that the congruences (7.4) and (7.5) are derived. The fact that Γ_k^i, Λ_k^i, Θ_k^i are circuits is a consequence of the fact that all elements of E_k^* subsequent to the r_kth column are zero.

The homologies (7.2) and (7.3) arise by similar reasoning from the matrix equation

$$E_{k+1} \cdot D_{k+1} = C_k \cdot E_{k+1}^* \tag{7.8}$$

in which it is to be remembered that the columns of C_k are the same as those of D_k in a different order.

8. The $(n-k)$-dimensional complexes required in the formulas of § 6 may be determined by the same process as described in § 7, from the matrices of the dual complex C_n'. The matrices of the dual complex are related to those of C_n by the equation (cf. p. 123).

$$\bar{E}_{n-k} = E_{k+1}' \tag{8.1}$$

in which $\bar{E}_{n-k}$ is the matrix of the relations between $(n-k-1)$-cells and $(n-k)$-cells of C_n' and E_{k+1}' is the matrix obtained by interchanging rows and columns of E_{k+1}. The equation (7.8) gives the following:

$$C_k^{-1} \cdot E_{k+1} \cdot D_{k+1} = E_{k+1}^*,$$

$$D_{k+1}' \cdot E_{k+1}' \cdot C_k^{-1'} = E_{k+1}^*,$$

$$\bar{E}_{n-k} \cdot C_k^{-1'} = \bar{E}_{n-k}^* \cdot D_{k+1}^{-1'}.$$

The columns of $C_k^{-1'}$ determine a linearly independent set of complexes analogous to those determined by the columns of D_k. They are described by the following homologies and congruences, written in the order of the columns of $C_k^{-1'}$:

(8.2) $$\bar{\Phi}^j_{n-k} \equiv \bar{\Theta}^j_{n-k-1}, \qquad 0 < j \leq r_{k+1} - \tau_k;$$

(8.3) $$\bar{\Psi}^j_{n-k} \equiv t_j^{n-k-1} \bar{\Lambda}^j_{n-k-1}, \qquad 0 < j \leq \tau_k;$$

(8.4) $$\bar{\Gamma}^j_{n-k} \equiv 0, \qquad 0 < j \leq P_k - 1;$$

(8.5) $$\bar{\Theta}^j_{n-k} \sim 0, \qquad 0 < j \leq r_k - \tau_{k-1};$$

(8.6) $$t_j^{n-k} \Lambda^j_{n-k} \sim 0, \qquad 0 < j \leq \tau_{k-1}.$$

9. Since the columns of D_k are the same as those of C_k in a different order, and the columns of $C_k^{-1'}$ are the same as the rows of C_k^{-1}, the matrix equation

(9.1) $$C_k^{-1} \cdot C_k = 1$$

implies the relations

(9.2) $$\sum_{i=1}^{a_k} x_{ij}\, x'_{ip} = \begin{cases} 1 \text{ if } j = p \\ 0 \text{ if } j \neq p \end{cases}$$

between the columns $(x_{1j}, x_{2j}, \ldots, x_{a_k j})$ of D_k and the columns $(x'_{1p}, x'_{2p}, \ldots, x'_{a_k p})$ of C_k^{-1}. But by (5.3) this implies that the intersection numbers of Γ^j_k, Λ^j_k, etc., with $\bar{\Gamma}^j_{n-k}$, $\bar{\Lambda}^j_{n-k}$, etc., are zero except in the following α_k cases, written in the order of the columns of C_k^{-1}:

(9.3) $$N(\Theta^j_k, \bar{\Phi}^j_{n-k}) = (-1)^{k(k+1)/2}, \qquad 0 < j \leq r_{k+1} - \tau_k;$$

(9.4) $$N(\Lambda^j_k, \bar{\Psi}^j_{n-k}) = (-1)^{k(k+1)/2}, \qquad 0 < j \leq \tau_k;$$

(9.5) $$N(\Gamma^j_k, \bar{\Gamma}^j_{n-k}) = (-1)^{k(k+1)/2}, \qquad 0 < j \leq P_k - 1;$$

(9.6) $$N(\Phi^j_k, \Theta^j_{n-k}) = (-1)^{k(k+1)/2}, \qquad 0 < j \leq r_k - \tau_{k-1};$$

(9.7) $$N(\Psi^j_k, \Lambda^j_{n-k}) = (-1)^{k(k+1)/2}, \qquad 0 < j \leq \tau_{k-1}.$$

Thus, each k-circuit Γ_k^i intersects the corresponding $(n-k)$-circuit once and intersects no other of the fundamental $(n-k)$-dimensional complexes. None of the other k-circuits ($\varLambda_k^i$ or Θ_k^i) intersects any $(n-k)$-circuits, but each Θ_k^i intersects a complex Φ_{n-k}^i which is bounded by Θ_{n-k-1}^i; and each $\varLambda_k^i$ intersects a complex Ψ_{n-k}^i which is bounded by $\varLambda_{n-k-1}^i$ counted t_i^k times. Thus we may say that each Θ_k^i *links* one and only one Θ_{n-k-1}^i once and each $\varLambda_k^i$ links one $\varLambda_{n-k-1}^i$ in a manner which may be described as a fractional number of times, $\pm 1/t_i^k$. A further study of these linkages would carry us beyond the bounds of the present paper.

10. The matrix spoken of at the end of § 6 is now seen to consist entirely of zeros except for α_k elements whose value, 1 in every case, is given by equations (9.3), ..., (9.7). If we limit attention to circuits the only non-zero terms which remain are those given by the intersections of Γ_k', ..., $\Gamma_k^{P_1-1}$ with the corresponding non-bounding $(n-k)$-circuits. The matrix is therefore one which consists entirely of zeros except for the first P_1-1 terms of the main diagonal which are all 1's. For any k-circuit Γ_k of C_n we have

$$(10.1) \qquad \Gamma_k = \sum_{i=1}^{P_k-1} x_i \Gamma_k^i + \sum_{i=1}^{\tau_k} y_i \varLambda_k^i + \sum_{i=1}^{r_{k-1}-\tau_k} z_i \Theta_k^i$$

and for any $(n-k)$-circuit of C_n' we have

$$(10.2) \qquad \overline{\Gamma}_{n-k} = \sum_{i=1}^{P_k-1} x_i' \Gamma_{n-k}^i + \sum_{i=1}^{\tau_k} y_i' \varLambda_k^i + \sum_{i=1}^{r_{k+1}-\tau_k} z_i' \Theta_k^i.$$

When these expressions are substituted in (6.5) there results

$$(10.3) \qquad N(\Gamma_k, \overline{\Gamma}_{n-k}) = (-1)^{k(k+1)/2} \sum_{i=1}^{P_k-1} x_i x_i'.$$

Thus we have the theorem that *if*

$$(10.4) \qquad \Gamma_k \sim \sum_{i=1}^{P_k-1} x_i \Gamma_k^i + \sum_{i=1}^{\tau_k} y_i \varLambda_k^i$$

and

$$(10.5) \qquad \Gamma_{n-k} \sim \sum_{i=1}^{P_k-1} x_i' \overline{\Gamma}_{n-k}^i + \sum_{i=1}^{\tau_k} y_i' \overline{\varLambda}_{n-k}^i,$$

then the intersection number of Γ_k with Γ_{n-k} is given by (10.3).

This theorem has the corollary that

$$N(\Gamma_k, \Gamma_{n-k}) = 0$$

if and only if at least one of the homologies

$$p\,\Gamma_k \sim 0 \quad or \quad q\,\Gamma_{n-k} \sim 0$$

is satisfied for some integer value of p or q. In other words, the statement $p\,\Gamma_k \sim 0$ is equivalent to the equation

$$N(\Gamma_k, \Gamma_{n-k}) = 0$$

for the one circuit Γ_k and all circuits Γ_{n-k}.

From this it follows that if Γ_k' is any k-circuit composed of cells of C_n and such that

$$\Gamma_k \sim \Gamma_k'$$

then

$$N(\Gamma_k, \Gamma_{n-k}) = N(\Gamma_k', \Gamma_{n-k}).$$

11. Incidentally it may be remarked that (10.4) and (10.5) give rise to the following "homologies with division allowed":

$$\Gamma_k \sim \sum_{i=1}^{P_k-1} x_i\,\Gamma_k^i, \qquad \bar{\Gamma}_{n-k} \sim \sum_{i=1}^{P_k-1} y_i\,\bar{\Gamma}_{n-k}^i.$$

Whenever these homologies are satisfied the equation (10.3) is satisfied. As remarked by Poincaré, it is because the intersection numbers are more closely related to the homologies with division allowed than to the ordinary homologies that his attempt to prove the Euler theorem and the theorem about the duality of the Betti numbers by means of the intersection numbers was unsuccessful.

12. The fundamental sets of circuits which appear in the formulas of § 10 are chosen in a very special manner. A perfectly arbitrary fundamental set of k-circuits is however related to this special set by homologies

$$\Gamma_k^i \sim \sum_{j=1}^{P_k-1} \alpha_j^i \, \Gamma_k^j + \sum_{j=P_k}^{P_k-1+\tau_k} \alpha_j^i \, \Delta_k^{j-P_k+1}$$

in which the $(P_k-1+\tau_k)$-rowed determinant $|\alpha_j^i| = \pm 1$. A general fundamental set of $(n-k)$-circuits $\overline{\Gamma}_{n-k}^i$ is related to the special set by an analogous set of homologies. Hence the matrix of the intersection numbers

$$N(\overline{\Gamma}_k^i, \overline{\Gamma}_{n-k}^j)$$

is one of $P_k-1+\tau_k$ rows and $P_k-1+\tau_{k-1}$ columns, of rank P_k-1 and having all its invariant factors unity.

13. For some purposes it is desirable to introduce intersection numbers which do not distinguish between positive and negative intersections. The theory of these numbers is much simpler than that which we have been developing because all the determinations of algebraic sign in §§ 2, 3, 4, 5 can be omitted. We simply replace the definitions of § 3 by the agreement that

$$M(a_i^k, b_j^{n-k}) = 1 \text{ or } 0$$

according as a_i^k and b_j^{n-k} have a common point or not. Then the definition in § 5 is replaced by

$$M(\Gamma_k, \Gamma'_{n-k}) = \sum_{j=1}^{\alpha_k} x^j \, y^j,$$

the sum being taken modulo 2.

The determination of the intersection numbers of fundamental sets of k-circuits and $(n-k)$-circuits in §§ 7, 8, 9 is replaced by an analogous theory based on the matrices A_{k-1} and B_k which arise in the reduction of the incidence matrix H_k to normal form (cf. p. 79 and following pages). The result obtained is that there exist a set of k-circuits $\Gamma_k^1, \Gamma_k^2, \ldots, \Gamma_k^{R_k-1}$ and a set of $(n-k)$-circuits $\Gamma_{n-k}^1, \Gamma_{n-k}^2, \ldots, \Gamma_{n-k}^{R_k-1}$ such that

$$M(\Gamma_k^i, \Gamma_{n-k}^j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases};$$

and if

$$\Gamma_k = \sum_{i=1}^{R_k-1} x_i \, \Gamma_k^i,$$

$$\Gamma_{n-k} = \sum_{i=1}^{R_k-1} y_i \, \Gamma_{n-k}^i .$$

then

$$M(\Gamma_k, \Gamma_{n-k}) = \sum_{i=1}^{R_k-1} x_i \, y_i \qquad (\text{mod } 2).$$

It should be observed that these formulas cannot be obtained by reducing the formulas of § 10, modulo 2, because the formulas of the present section take account of non-orientable circuits which do not enter into the theory of oriented intersections.

PRINCETON UNIVERSITY,
PRINCETON, N. J.

THE GEOMETRY OF PATHS*

BY

OSWALD VEBLEN AND TRACY YERKES THOMAS

1. Introduction. The first part of this paper is intended as a systematic general account of the geometry of paths and is largely based on the series of notes by Eisenhart and Veblen in volume 8 of the Proceedings of the National Academy of Sciences. The general theory is carried far enough to include an account of a series of tensors defined by means of normal coördinates, and also a series of generalizations of the operation of covariant differentiation. We then turn to a special problem, the investigation of the conditions which must be satisfied by the functions Γ in order that the differential equations of the paths shall possess homogeneous first integrals.† We first solve a still more special problem for first integrals of the nth degree (§ 15). This includes as a special case the problem solved by Eisenhart and Veblen in the Proceedings of the National Academy of Sciences, vol. 8 (1922), p. 19, of finding the conditions which must be satisfied by the Γ's in order that the equations (2.1) shall be the differential equations of the geodesics of a Riemann space.

Finally we solve the general problem for the linear and quadratic cases; that is to say, we find algebraic necessary and sufficient conditions on the functions Γ in order that (2.1) shall possess homogeneous linear and quadratic first integrals. The method used will generalize to homogeneous first integrals of the nth degree. We leave unsolved all the projective problems which correspond to the affine problems which we have solved. For example, the problem remains open to find what condition must be satisfied by the Γ's in order that one of the sets of differential equations which define the same paths as (2.1) shall have a linear first integral.

2. The geometry of paths. Consider an n-dimensional region the points of which can be represented by coördinates $(x^1, x^2, \ldots, x^n)$. Also consider a set of differential equations

* Presented to the Society, October 28, 1922, and April 28, 1923.

† Our problem is distinguished from the problem of the existence of first integrals in dynamical systems (studied by Staeckel, Painlevé, Levi-Civita, and others) by the fact that the dynamical problem presupposes the existence of the integral corresponding to the fundamental quadratic form. Cf. Ricci and Levi-Civita, *Méthodes de calcul différentiel absolu,* Mathematische Annalen, vol. 54 (1901), p. 125.

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\,\frac{dx^\beta}{ds} = 0 \tag{2.1}$$

in which

$$\Gamma^i_{jk} = \Gamma^i_{kj}. \tag{2.2}$$

In these expressions the subscripts and superscripts take all integer values from 1 to n and the convention is employed that any term which contains the same index twice, once as a subscript and once as a superscript, represents a summation with respect to every such index. Thus the second term represents a quadratic form in dx^i/ds. The coefficients are arbitrary analytic functions of $(x^1, x^2, \ldots, x^n)$. The condition (2.2) is no restriction on the differential equations (2.1) because the coefficients of any quadratic form can be written so as to satisfy (2.2).

Any curve

$$x^i = \psi^i(s) \tag{2.3}$$

which satisfies (2.1) is called a *path* and the theory of these paths is what we call the geometry of paths.

The geometry of paths is a natural generalization of the euclidean geometry. For the differential equations of the straight lines in an n-dimensional euclidean space are

$$\frac{d^2x^i}{ds^2} = 0 \tag{2.4}$$

when referred to a cartesian coördinate system. An arbitrary transformation of the coördinates

$$x^i = \varphi^i(y^1, y^2, \ldots, y^n) \tag{2.5}$$

transforms (2.4) into a set of differential equations of the form (2.1) in the variables y, in which

$$\Gamma^i_{jk} = \frac{\partial y^i}{\partial x^\alpha}\,\frac{\partial^2 x^\alpha}{\partial y^j\,\partial y^k}. \tag{2.6}$$

Hence the system of paths defined by (2.1) has the properties of the straight lines of euclidean space whenever the functions Γ are such that (2.1) can be transformed by an analytic transformation into (2.4). This transformation is possible if and only if

$$\frac{\partial \Gamma^i_{jk}}{\partial x^l} - \frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^\alpha_{jk}\,\Gamma^i_{\alpha l} - \Gamma^\alpha_{jl}\,\Gamma^i_{\alpha k} = 0, \tag{2.7}$$

as can easily be proved. The left member of this equation is denoted by B^i_{jkl} and is called *the curvature tensor*.

The paths defined by (2.1) are the geodesics of a Riemann space in case the Γ's are such that there exists a quadratic differential form

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{2.8}$$

such that

$$\frac{\partial g_{ij}}{\partial x^k} - g_{i\alpha}\, \Gamma^\alpha_{jk} - g_{j\alpha}\, \Gamma^\alpha_{ik} = 0. \tag{2.9}$$

In this case the paths are the geodesics of the differential form (2.8).

The geometry of paths reduces to a Weyl metric geometry if the Γ's are such that there exists a linear form $\varphi_\alpha\, dx^\alpha$ and a quadratic form $g_{\alpha\beta}\, dx^\alpha\, dx^\beta$ such that

$$g_{\alpha k}\, \Gamma^\alpha_{ij} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) + \frac{1}{2}\left(g_{ik}\, \varphi_j + g_{jk}\, \varphi_i - g_{ij}\, \varphi_k\right) \tag{2.10}$$

(cf. H. Weyl, *Raum, Zeit, Materie,* 4th edition, p. 113).

In the general case (no restriction on the Γ's except (2.2)) the geometry of paths is equivalent to the geometry of infinitesimal parallelism as developed by Weyl, in *Raum, Zeit, Materie* (4th edition, p. 100). For any system of Γ's which appear in the differential equations (2.1) can be used to establish a definition of infinitesimal parallelism according to which the paths defined by (2.1) are geodesics in the sense of Weyl.

3. **Transformation of the dependent variables.** Consider an arbitrary analytic transformation of the coördinates

$$\bar{x}^i = f^i(x^1, x^2, \ldots, x^n) \tag{3.1}$$

which may also be written

$$x^i = g^i(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n). \tag{3.2}$$

By substituting (2.3) in (3.1) we obtain

$$\bar{x}^i = \psi^i(s) \tag{3.3}$$

as the equation of the path represented by (2.3). Since

$$\frac{dx^i}{ds} = \frac{\partial x^i}{\partial \bar{x}^\alpha}\, \frac{d\bar{x}^\alpha}{ds} \tag{3.4}$$

and

$$\frac{d^2 x^i}{ds^2} = \frac{\partial^2 x^i}{\partial \bar{x}^\alpha \partial \bar{x}^\beta} \frac{d\bar{x}^\alpha}{ds} \frac{d\bar{x}^\beta}{ds} + \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{d^2 \bar{x}^\alpha}{ds^2} \tag{3.5}$$

we find that this path satisfies the differential equation

$$\frac{d^2 \bar{x}^i}{ds^2} + \bar{\Gamma}^i_{\alpha\beta} \frac{d\bar{x}^\alpha}{ds} \frac{d\bar{x}^\beta}{ds} = 0, \tag{3.6}$$

in which

$$\bar{\Gamma}^i_{jk} = \frac{\partial \bar{x}^i}{\partial x^\alpha} \left(\frac{\partial^2 x^\alpha}{\partial \bar{x}^j \partial \bar{x}^k} + \Gamma^\alpha_{\beta\gamma} \frac{\partial x^\beta}{\partial \bar{x}^j} \frac{\partial x^\gamma}{\partial \bar{x}^k} \right). \tag{3.7}$$

Thus the form of the equation (2.1) persists under a transformation of coördinates. It follows from (3.7) that the functions Γ behave like the components of a tensor under linear transformations with constant coefficients but not under more general transformations. It is seen by an easy computation that the functions B^i_{jkl} defined in § 2 are the components of a tensor. It follows at once that the equation

$$S_{kl} = B^i_{ikl} = \frac{\partial \Gamma^i_{ik}}{\partial x^l} - \frac{\partial \Gamma^i_{il}}{\partial x^k} \tag{3.8}$$

defines a tensor which is skew symmetric. This tensor is identically zero in the Riemann geometry. It also follows that

$$R_{jk} = B^i_{jki} = \frac{\partial \Gamma^i_{jk}}{\partial x^i} - \frac{\partial \Gamma^i_{ji}}{\partial x^k} + \Gamma^\alpha_{jk} \Gamma^i_{\alpha i} - \Gamma^\alpha_{ji} \Gamma^i_{\alpha k} \tag{3.9}$$

is a tensor. This we shall call the Ricci tensor because it reduces to the tensor studied by Ricci* for the case of the Riemann geometry. It is symmetric if and only if $S_{ij} = 0$, as is obvious on comparing (3.8) and (3.9). Further properties of these tensors are to be found in a paper by Eisenhart in the Annals of Mathematics (vol. 24).

For convenience of reference we put down here the following formulas about transformation of coördinates in general:

---

* G. Ricci, Atti, Reale Istituto Veneto, vol. 63 (1903), pp. 1233–1239.

$$\frac{\partial \bar{x}^i}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial \bar{x}^j} = \delta^i_j; \tag{3.10}$$

$$\frac{\partial^2 \bar{x}^i}{\partial x^\alpha \partial x^\beta}\frac{\partial x^\alpha}{\partial \bar{x}^j} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial \bar{x}^i}{\partial x^\alpha}\frac{\partial \bar{x}^k}{\partial x^\beta} = 0; \tag{3.11}$$

$$\frac{\partial^2 \bar{x}^i}{\partial x^\alpha \partial x^\beta}\frac{\partial x^\alpha}{\partial \bar{x}^j}\frac{\partial x^\beta}{\partial \bar{x}^k} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial \bar{x}^i}{\partial x^\alpha} = 0; \tag{3.12}$$

$$\begin{aligned} &\frac{\partial^3 \bar{x}^i}{\partial x^\alpha \partial x^\beta \partial x^\gamma}\frac{\partial x^\alpha}{\partial \bar{x}^j} + \frac{\partial^2 \bar{x}^i}{\partial x^\alpha \partial x^\beta}\frac{\partial^2 x^\alpha}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial \bar{x}^k}{\partial x^\gamma} + \frac{\partial^3 x^\alpha}{\partial \bar{x}^j \partial \bar{x}^k \partial \bar{x}^l}\frac{\partial \bar{x}^i}{\partial x^\alpha}\frac{\partial \bar{x}^k}{\partial x^\beta}\frac{\partial \bar{x}^l}{\partial x^\gamma} \\ &\quad + \frac{\partial^2 x^\alpha}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial^2 \bar{x}^i}{\partial x^\alpha \partial x^\gamma}\frac{\partial \bar{x}^k}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial \bar{x}^i}{\partial x^\alpha}\frac{\partial^2 \bar{x}^k}{\partial x^\beta \partial x^\gamma} = 0; \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\frac{\partial^3 \bar{x}^i}{\partial x^\alpha \partial x^\beta \partial x^\gamma}\frac{\partial x^\alpha}{\partial \bar{x}^j}\frac{\partial x^\gamma}{\partial \bar{x}^l} + \frac{\partial^2 \bar{x}^i}{\partial x^\alpha \partial x^\beta}\frac{\partial^2 x^\alpha}{\partial \bar{x}^j \partial \bar{x}^l} + \frac{\partial^3 x^\alpha}{\partial \bar{x}^j \partial \bar{x}^k \partial \bar{x}^l}\frac{\partial \bar{x}^i}{\partial x^\alpha}\frac{\partial \bar{x}^k}{\partial x^\beta} \\ &\quad + \frac{\partial^2 x^\alpha}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial^2 \bar{x}^i}{\partial x^\alpha \partial x^\gamma}\frac{\partial \bar{x}^k}{\partial x^\beta}\frac{\partial x^\gamma}{\partial \bar{x}^l} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial^2 \bar{x}^k}{\partial x^\beta \partial x^\gamma}\frac{\partial \bar{x}^i}{\partial x^\alpha}\frac{\partial x^\gamma}{\partial \bar{x}^l} = 0; \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\frac{\partial^3 \bar{x}^i}{\partial x^\alpha \partial x^\beta \partial x^\gamma}\frac{\partial x^\alpha}{\partial \bar{x}^j}\frac{\partial x^\beta}{\partial \bar{x}^k} + \frac{\partial^2 \bar{x}^i}{\partial x^\alpha \partial x^\beta}\frac{\partial^2 x^\alpha}{\partial \bar{x}^j \partial \bar{x}^l}\frac{\partial x^\beta}{\partial \bar{x}^k}\frac{\partial \bar{x}^l}{\partial x^\gamma} \\ &\quad + \frac{\partial^2 \bar{x}^i}{\partial x^\alpha \partial x^\beta}\frac{\partial^2 x^\beta}{\partial \bar{x}^k \partial \bar{x}^l}\frac{\partial x^\alpha}{\partial \bar{x}^j}\frac{\partial \bar{x}^l}{\partial x^\gamma} + \frac{\partial^3 x^\alpha}{\partial \bar{x}^j \partial \bar{x}^k \partial \bar{x}^l}\frac{\partial \bar{x}^i}{\partial x^\alpha}\frac{\partial \bar{x}^l}{\partial x^\gamma} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial^2 \bar{x}^i}{\partial x^\alpha \partial x^\gamma} = 0; \end{aligned} \tag{3.15}$$

$$\begin{aligned} &\frac{\partial^3 \bar{x}^i}{\partial x^\alpha \partial x^\beta \partial x^\gamma}\frac{\partial x^\alpha}{\partial \bar{x}^j}\frac{\partial x^\beta}{\partial \bar{x}^k}\frac{\partial x^\gamma}{\partial \bar{x}^l} + \frac{\partial^2 \bar{x}^i}{\partial x^\alpha \partial x^\beta}\frac{\partial^2 x^\alpha}{\partial \bar{x}^j \partial \bar{x}^l}\frac{\partial x^\beta}{\partial \bar{x}^k} \\ &\quad + \frac{\partial^2 \bar{x}^i}{\partial x^\alpha \partial x^\beta}\frac{\partial^2 x^\beta}{\partial \bar{x}^k \partial \bar{x}^l}\frac{\partial x^\alpha}{\partial \bar{x}^j} + \frac{\partial^3 x^\alpha}{\partial \bar{x}^j \partial \bar{x}^k \partial \bar{x}^l}\frac{\partial \bar{x}^i}{\partial x^\alpha} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial^2 \bar{x}^i}{\partial x^\alpha \partial x^\beta}\frac{\partial x^\beta}{\partial \bar{x}^l} = 0. \end{aligned} \tag{3.16}$$

4. **Transformation of the independent variable.** If we make an arbitrary analytic substitution

$$s = f(t), \tag{4.1}$$

$$t = g(s). \tag{4.2}$$

in the equation of a path (2.3) the latter becomes

$$x^i = \varphi^i(t). \tag{4.3}$$

For this path we have

$$\frac{d^2x^i}{ds^2} = \frac{d^2x^i}{dt^2}\left(\frac{dt}{ds}\right)^2 + \frac{dx^i}{dt}\frac{d^2t}{ds^2}. \tag{4.4}$$

On comparison with (2.1) we see that the equation (4.3) satisfies the differential equation

$$\frac{\dfrac{d^2x^i}{dt^2} + \Gamma^i_{\alpha\beta}\dfrac{dx^\alpha}{dt}\dfrac{dx^\beta}{dt}}{\dfrac{dx^i}{dt}} = -\frac{\dfrac{d^2t}{ds^2}}{\left(\dfrac{dt}{ds}\right)^2}. \tag{4.5}$$

Hence the differential equations

$$\frac{\dfrac{d^2x^i}{dt^2} + \Gamma^i_{\alpha\beta}\dfrac{dx^\alpha}{dt}\dfrac{dx^\beta}{dt}}{\dfrac{dx^i}{dt}} = \frac{\dfrac{d^2x^j}{dt^2} + \Gamma^j_{\alpha\beta}\dfrac{dx^\alpha}{dt}\dfrac{dx^\beta}{dt}}{\dfrac{dx^j}{dt}} \tag{4.6}$$

are satisfied by the equations of the paths and are such that they continue to be satisfied if the independent variable in the parameter representation (4.3) of any path is subjected to an arbitrary transformation.

From (4.5) it is evident that the differential equations (2.1) will continue to be satisfied if the independent variable in the equations of a path (2.3) be replaced by t where

$$t = as + b, \tag{4.7}$$

a and b being constants.

The differential equations (4.6) are due to J. L. Synge, who has pointed out that the system of paths defined by them is no more general than that defined by (2.1). For, suppose that (4.3) satisfies (4.6). Let $\Phi(t)$ be the function of t obtained by substituting (4.3) in any of the expressions whose equality is asserted by (4.6). The following equation is satisfied by (4.3):

$$\frac{\dfrac{d^2x^i}{dt^2} + \Gamma^i_{\alpha\beta}\dfrac{dx^\alpha}{dt}\dfrac{dx^\beta}{dt}}{\dfrac{dx^i}{dt}} = \Phi(t). \tag{4.8}$$

Substitute (4.2) in (4.3), obtaining

$$x^i = \psi^i(s),$$

an equation of the paths which must satisfy

$$\frac{\dfrac{d^2x^i}{ds^2} + \Gamma^i_{\alpha\beta}\dfrac{dx^\alpha}{ds}\dfrac{dx^\beta}{ds}}{\dfrac{dx^i}{ds}} = \frac{\Phi(t)}{\dfrac{ds}{dt}} - \frac{\dfrac{d^2s}{dt^2}}{\left(\dfrac{ds}{dt}\right)^2}. \tag{4.9}$$

Now if

$$s = f(t) = A + B\int e^{\int\Phi(t)dt}dt, \tag{4.10}$$

A and B being constants, (4.9) reduces to (2.1). Hence the equations of any path defined by (4.6) may be written as solutions of (2.1).

5. **Projective geometry of paths.*** Let us inquire under what circumstances a set of differential equations

$$\frac{d^2x^i}{ds^2} + \Lambda^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \tag{5.1}$$

can represent the same system of paths as (2.1). Suppose that a curve

$$x^i = \varphi^i(t) \tag{5.2}$$

is a path both for (5.1) and for (2.1). The functions $\varphi^i(t)$ are not necessarily solutions of (2.1) or of (5.1), but they are solutions of Synge's equations (4.6) and also of the corresponding equations determined by (5.1), i. e. of

$$\frac{dx^j}{dt}\left(\frac{d^2x^i}{dt^2} + \Lambda^i_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\right) = \frac{dx^i}{dt}\left(\frac{d^2x^j}{dt^2} + \Lambda^j_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\right). \tag{5.3}$$

* The discovery that the same system of paths arises from (5.1) as from (2.1) when (5.5) and (5.8) are satisfied is due to Weyl, Göttinger Nachrichten, 1921, p. 99. See also Eisenhart, Proceedings of the National Academy of Sciences, vol. 8 (1922), p. 233, and Veblen, ibid., p. 347. In the latter paper in equation (2.6) the final t should be omitted and $dx^1/dt, \ldots, dx^n/dt$ should be evaluated at the point q; also the integration signs are missing in (4.2).

Between (5.3) and (4.6) we can eliminate the second derivatives, thus obtaining

$$\frac{(\Gamma^i_{\alpha\beta} - \Lambda^i_{\alpha\beta})}{\frac{dx^i}{dt}} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = \frac{(\Gamma^j_{\alpha\beta} - \Lambda^j_{\alpha\beta})}{\frac{dx^j}{dt}} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt}. \tag{5.4}$$

Let

$$\Gamma^i_{\alpha\beta} - \Lambda^i_{\alpha\beta} = \Phi^i_{\alpha\beta} \tag{5.5}$$

and

$$\frac{1}{n+1} \Phi^i_{i\beta} = \Phi_\beta. \tag{5.6}$$

If we subtract from (3.7) the corresponding equations for the functions Λ the result shows that $\Phi^i_{\alpha\beta}$ is a tensor. Hence Φ_β is a vector. The equation (5.4) now becomes

$$\left(\Phi^i_{\alpha\beta} \frac{dx^j}{dt} - \Phi^j_{\alpha\beta} \frac{dx^i}{dt}\right) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0.$$

In this we put

$$\frac{dx^i}{dt} = \delta^i_\gamma \frac{dx^\gamma}{dt} \quad \text{and} \quad \frac{dx^j}{dt} = \delta^j_\gamma \frac{dx^\gamma}{dt},$$

and obtain

$$(\Phi^i_{\alpha\beta} \delta^j_\gamma - \Phi^j_{\alpha\beta} \delta^i_\gamma) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \frac{dx^\gamma}{dt} = 0. \tag{5.7}$$

Since the derivatives dx^α/dt may be chosen arbitrarily this gives

$$\Phi^i_{\alpha\beta} \delta^j_\gamma + \Phi^i_{\beta\gamma} \delta^j_\alpha + \Phi^i_{\gamma\alpha} \delta^j_\beta = \Phi^j_{\alpha\beta} \delta^i_\gamma + \Phi^j_{\beta\gamma} \delta^i_\alpha + \Phi^j_{\gamma\alpha} \delta^i_\beta.$$

If we set $j = \gamma$ in this equation and sum with respect to γ, we obtain

$$n \Phi^i_{\alpha\beta} + \Phi^i_{\beta\alpha} + \Phi^i_{\beta\alpha} = \Phi^i_{\alpha\beta} + (n+1) \Phi_\beta \delta^i_\alpha + (n+1) \Phi_\alpha \delta^i_\beta.$$

Hence

$$\Phi^i_{\alpha\beta} = \Phi_\alpha \delta^i_\beta + \Phi_\beta \delta^i_\alpha. \tag{5.8}$$

Hence, if the equations (5.1) and (2.1) are to determine the same system of paths, the functions Γ and Λ must be related by (5.5) and (5.8).

Conversely, let Φ_α be any covariant vector and let the tensor $\Phi^i_{\alpha\beta}$ be defined by the equations (5.8). Then any two sets of differential equations (2.1) and (5.1) will define the same system of paths, provided that (5.5) is satisfied. For consider any path with respect to the Γ's. Along this path we have

$$2\,\Phi_\alpha \frac{dx^\alpha}{dt} = \frac{\Phi^i_{\alpha\beta}\dfrac{dx^\alpha}{dt}\dfrac{dx^\beta}{dt}}{\dfrac{dx^i}{dt}} = \frac{(\Gamma^i_{\alpha\beta}-\Lambda^i_{\alpha\beta})}{\dfrac{dx^i}{dt}}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}.$$

Hence (5.4) is satisfied. But if (5.4) is added to (4.6) the corresponding equations in Λ are obtained. Hence every path with respect to the Γ's is also a path with respect to the Λ's.

A system of functions $\Gamma^i_{\alpha\beta}$ determines a definition of infinitesimal parallelism in the sense of Levi-Civita and Weyl. It is therefore appropriate to designate the body of theorems which state those properties which are determined by a particular set of differential equations (2.1) as an *affine geometry of paths.* In like manner the body of theorems which state properties of a system of paths independently of any particular definition of affine connection (i. e. of any particular set of differential equations (2.1)) may be called a *projective geometry of paths.*

For example, the theory of the curvature tensor belongs to the affine geometry of paths. For if the curvature tensor determined by (2.1) is denoted by $B^i_{\alpha\beta\gamma}$ as in § 2, the corresponding curvature tensor determined by (5.1) is

$$(5.9)\quad B^i_{\alpha\beta\gamma} - \delta^i_\alpha\,\Phi_{\beta,\gamma} + \delta^i_\alpha\,\Phi_{\gamma,\beta} - \delta^i_\beta\,\Phi_{\alpha,\gamma} + \delta^i_\gamma\,\Phi_{\alpha,\beta} - \delta^i_\beta\,\Phi_\alpha\,\Phi_\gamma + \delta^i_\gamma\,\Phi_\alpha\,\Phi_\beta.$$

In this expression $\Phi_{\alpha,\beta}$ denotes the covariant derivative (cf. § 10 below) of Φ_α with respect to the functions $\Gamma^i_{\alpha\beta}$.

The Ricci tensor $R_{\alpha\beta}$ becomes

$$(5.10)\qquad R_{\alpha\beta} + n\,\Phi_{\alpha,\beta} - \Phi_{\beta,\alpha} + (n-1)\,\Phi_\alpha\,\Phi_\beta,$$

and the skew symmetric tensor $S_{\alpha\beta}$ becomes

$$(5.11)\qquad S_{\alpha\beta} - (n+1)(\Phi_{\alpha,\beta} - \Phi_{\beta,\alpha}).$$

Comparing these three expressions, it is evident that a tensor which is the same for the Λ's as for the Γ's is defined as follows:

$$W^i_{\alpha\beta\gamma} = B^i_{\alpha\beta\gamma} - \frac{\delta^i_\gamma R_{\alpha\beta}}{n-1} + \frac{\delta^i_\beta R_{\alpha\gamma}}{n-1} - \frac{1}{n^2-1}(\delta^i_\gamma S_{\alpha\beta} - \delta^i_\beta S_{\alpha\gamma}) - \frac{\delta^i_\alpha S_{\beta\gamma}}{n+1}.$$

This is what Weyl (loc. cit.) calls the projective curvature tensor, and its theory belongs to the projective geometry of paths. It can also be written in the form

$$W^i_{\alpha\beta\gamma} = B^i_{\alpha\beta\gamma} + \frac{\delta^i_\beta}{n^2-1}(n R_{\alpha\gamma} + R_{\gamma\alpha})$$
$$- \frac{\delta^i_\gamma}{n^2-1}(n R_{\alpha\beta} + R_{\beta\alpha}) + \frac{\delta^i_\alpha}{n+1}(R_{\beta\gamma} - R_{\gamma\beta}).$$

In the rest of this paper we shall be concerned entirely with the affine geometry of paths, to which we now return.

6. **Equations of the paths.** A unique solution of (2.1) in the form (2.3) can be found which satisfies a set of initial conditions

$$q^i = \psi^i(0), \tag{6.1}$$

$$\xi^i = \frac{d}{ds}\psi^i(0), \tag{6.2}$$

where $q^1, q^2, \ldots, q^n$ and $\xi^1, \xi^2, \ldots, \xi^n$ are arbitrary constants. For if we differentiate (2.1) successively we obtain the following sequence of equations:

$$\begin{aligned} &\frac{d^2x^i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0, \\ &\frac{d^3x^i}{ds^3} + \Gamma^i_{\alpha\beta\gamma}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0, \\ &\frac{d^4x^i}{ds^4} + \Gamma^i_{\alpha\beta\gamma\delta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}\frac{dx^\delta}{ds} = 0, \\ &\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{6.3}$$

in which

$$(6.4)\quad \Gamma^i_{jkl} = \frac{1}{3}P\left(\frac{\partial \Gamma^i_{jk}}{\partial x^l} - \Gamma^i_{\alpha k}\Gamma^\alpha_{jl} - \Gamma^i_{j\alpha}\Gamma^\alpha_{kl}\right) = \frac{1}{3}P\left(\frac{\partial \Gamma^i_{jk}}{\partial x^l} - 2\Gamma^i_{\alpha j}\Gamma^\alpha_{kl}\right)$$

and, in general,

$$(6.5)\quad \begin{aligned} \Gamma^i_{jkl\ldots mn} &= \frac{1}{N}P\left(\frac{\partial \Gamma^i_{jkl\ldots m}}{\partial x^n} - \Gamma^i_{\alpha kl\ldots m}\Gamma^\alpha_{jn} - \cdots - \Gamma^i_{jkl\ldots\alpha}\Gamma^\alpha_{mn}\right) \\ &= \frac{1}{N}P\left[\frac{\partial \Gamma^i_{jkl\ldots m}}{\partial x^n} - (N-1)\,\Gamma^i_{\alpha jk\ldots l}\Gamma^\alpha_{mn}\right] \end{aligned}$$

where N denotes the number of subscripts, and the symbol P denotes the sum of the terms obtainable from the ones inside the parenthesis by permuting the set of subscripts cyclically. Thus the functions $\Gamma^i_{jkl\ldots mn}$ have the property of being unchanged by any permutation of the subscripts.* The equations (6.1), (6.2), (6.3) determine immediately the following series for ψ^i in terms of s:

$$(6.6)\quad x^i = q^i + \xi^i s - \frac{1}{2!}\Gamma^i_{\alpha\beta}(q)\,\xi^\alpha\xi^\beta s^2 - \frac{1}{3!}\Gamma^i_{\alpha\beta\gamma}(q)\,\xi^\alpha\xi^\beta\xi^\gamma s^3 - \cdots.$$

In this expression $\Gamma^i_{\alpha\beta\ldots\mu}(q)$ represents the value of $\Gamma^i_{\alpha\beta\ldots\mu}$ obtained by giving x^i the value q^i. In general we shall use x to represent $(x^1, x^2, \ldots, x^n)$, ξ to represent $(\xi^1, \xi^2, \ldots, \xi^n)$, and so on. For any point q and any "direction" ξ we have a unique path determined by (6.6). These equations may be abbreviated in the form

$$(6.7)\quad x^i = q^i + \Gamma^i(q, \xi s).$$

The jacobian of the equations (6.7) is equal to unity. Hence for values of x sufficiently near to q the equations can be solved, giving

$$(6.8)\quad \xi^i s = x^i - q^i + A^i(q, x-q),$$

where A^i is a multiple power series in $(x^i - q^i)$, beginning with second order terms. Hence there is one and only one path joining q to x.

---

* We have changed the notation used by Veblen in the Proceedings of the National Academy of Sciences, vol. 8 (1922), p. 192, in order to introduce this symmetry.

7. **Normal coördinates.** Let us now put

$$y^i = \xi^i s. \tag{7.1}$$

The equations (6.7) and (6.8) become

$$x^i = q^i + \Gamma^i(q, y) \tag{7.2}$$

and

$$y^i = x^i - q^i + \Delta^i(q, x-q). \tag{7.3}$$

These equations may be regarded as defining a transformation from the coördinates $(x^1, x^2, \ldots, x^n)$ to a new set of coördinates $(y^1, y^2, \ldots, y^n)$ which we shall call *normal coördinates* because they reduce to Riemann's normal coördinates in case the geometry of paths reduces to a Riemann geometry. This transformation changes the differential equations of the paths (2.1) into

$$\frac{d^2 y^i}{ds^2} + C^i_{\alpha\beta} \frac{dy^\alpha}{ds} \frac{dy^\beta}{ds} = 0, \tag{7.4}$$

where $C^i_{\alpha\beta}$ are functions of y defined by the equations

$$C^\alpha_{jk} \frac{\partial x^i}{\partial y^\alpha} = \frac{\partial^2 x^i}{\partial y^j \partial y^k} + \Gamma^i_{\beta\gamma} \frac{\partial x^\beta}{\partial y^j} \frac{\partial x^\gamma}{\partial y^k}. \tag{7.5}$$

These coördinates have been so chosen that the curves defined by (7.1) are the paths through the origin. If we take any point y there is one and only one of the paths (7.1) which passes through it. Substituting (7.1) in (7.4) we find

$$C^i_{\alpha\beta} \xi^\alpha \xi^\beta = 0, \tag{7.6}$$

and hence on multiplying by the square of the value of s determined for the point y by the equation (7.1) we obtain

$$C^i_{\alpha\beta} y^\alpha y^\beta = 0. \tag{7.7}$$

Let us now consider the effect of a transformation of the variables x of the form (3.1). This changes the equation of a path (2.3) which satisfies (2.1)

into the equation (3.3) which satisfies (3.6). It also changes the initial conditions (6.1) and (6.2) into

$$\bar{q}^i = \bar{\psi}^i(0) = f^i(q) \tag{7.8}$$

and

$$\bar{\xi}^i = \frac{d}{ds}\bar{\psi}^i(0) = \frac{d\psi^\alpha(0)}{ds}\left(\frac{\partial \bar{x}^i}{\partial x^\alpha}\right)_q \tag{7.9}$$

respectively, the subscript q indicating that the derivative is evaluated for $x = q$. For any point p, not too far away from q, there is a unique path and thus a unique set of values $(y^1, y^2, \ldots, y^n)$. From these we determine ξ^i and s so that $y^i = \xi^i s$. Then (3.3) gives the equation of the same path in terms of the coördinates $\bar{x}$ in such form that the point p is determined by the parameter s. Hence by (7.9),

$$\bar{y}^i = y^\alpha \left(\frac{\partial \bar{x}^i}{\partial x^\alpha}\right)_q. \tag{7.10}$$

In this formula the coefficients $\left(\frac{\partial \bar{x}^i}{\partial x^\alpha}\right)_q$ are independent of the particular path and dependent only on the point q and the two coördinate systems. Hence *when the coördinates x undergo an arbitrary analytic transformation, the normal coördinates determined by the coördinates x and a point q suffer a linear homogeneous transformation* (7.10) *with constant coefficients.* In other words the normal coördinates are transformed like contravariant vectors. They are not vectors, however, in the narrow sense, but are the components of a "step" from the origin of the normal coördinates to the point at which the coördinates are taken. An arbitrary step (AB) determined by the points A and B can be represented by the coördinates of the point B in the normal coördinate system associated with the point A.

8. **Alternative treatment of normal coördinates.** The identity (7.7) can be used as the definition of the normal coördinates. For by (3.12)

$$C^i_{\alpha\beta} = \left(\Gamma^\gamma_{jk}\frac{\partial y^i}{\partial x^\gamma} - \frac{\partial^2 y^i}{\partial x^j \partial x^k}\right)\frac{\partial x^j}{\partial y^\alpha}\frac{\partial x^k}{\partial y^\beta} \tag{8.1}$$

so that (7.7) becomes

$$\left(\Gamma^\gamma_{jk}\frac{\partial y^i}{\partial x^\gamma} - \frac{\partial^2 y^i}{\partial x^j \partial x^k}\right)\frac{\partial x^j}{\partial y^\alpha}\frac{\partial x^k}{\partial y^\beta} y^\alpha y^\beta = 0. \tag{8.2}$$

The differential equations (8.2) uniquely determine a functional relation between the x's and the y's when taken in conjunction with the initial conditions

$$y^i = 0 \text{ when } x^i = q^i, \tag{8.3}$$

$$\frac{\partial y^i}{\partial x^j} = \delta^i_j \text{ when } x^i = q^i. \tag{8.4}$$

For when we differentiate (8.2) repeatedly and substitute these initial conditions, making use of the formulas at the end of § 3, we find

$$\begin{aligned} x^i = q^i + y^i - \frac{1}{2!} \Gamma^i_{\alpha\beta}(q)\, y^\alpha y^\beta - \frac{1}{3!} \Gamma^i_{\alpha\beta\gamma}(q)\, y^\alpha y^\beta y^\gamma \\ - \frac{1}{4!} \Gamma^i_{\alpha\beta\gamma\delta}\, y^\alpha y^\beta y^\gamma y^\delta - \cdots \end{aligned} \tag{8.5}$$

and

$$\begin{aligned} y^i = x^i - q^i + \frac{1}{2!} A^i_{\alpha\beta}(q)\,(x^\alpha - q^\alpha)(x^\beta - q^\beta) \\ + \frac{1}{3!} A^i_{\alpha\beta\gamma}(x^\alpha - q^\alpha)(x^\beta - q^\beta)(x^\gamma - q^\gamma) + \cdots \end{aligned} \tag{8.6}$$

where the Γ's have the meaning given them in § 3 and the A's are such that

$$A^i_{jk} = \Gamma^i_{jk},$$

$$A^i_{jkl} = \Gamma^i_{jkl} + P(A^i_{\alpha j}\, \Gamma^\alpha_{kl}) = \frac{1}{3} P\left(\frac{\partial \Gamma^i_{jk}}{\partial x^l} - \Gamma^i_{\alpha j}\, \Gamma^\alpha_{kl}\right),$$

$$A^i_{jklm} = \Gamma^i_{jklm} + P(A^i_{\alpha j}\, \Gamma^\alpha_{klm} + A^i_{\alpha\beta}\, \Gamma^\alpha_{jk}\, \Gamma^\beta_{lm} + A^i_{\alpha jk}\, \Gamma^\alpha_{lm}),$$

. . . .

. . . .

. . . .

If the general solution of (8.2), regarded as a differential equation for y in terms of x, is denoted by $\bar{y}$ when only the initial conditions (8.3) are imposed, then

$$\bar{y}^i = a^i_\alpha\, y^\alpha \tag{8.7}$$

where the a^i_α are arbitrary constants, and y is the solution determined by the initial conditions (8.3) and (8.4) which is given by (8.6). This last theorem is proved by observing, first, that the function $\tilde{y}$ defined by (8.7) satisfies (8.2) and (8.3) and, second, that if there were any other solution for which

$$\tilde{y}^i = 0 \text{ and } \frac{\partial \tilde{y}^i}{\partial x^j} = a^i_j$$

when $x^i = q^i$, the solution (8.6) would not be uniquely determined by (8.4).

In order to show the tensor character of the normal coördinates let us now consider the effect of a transformation of the variables x of the form (3.1). We inquire what are the normal coördinates determined by $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n$. These normal coördinates which we shall denote by $\bar{y}^1, \bar{y}^2, \ldots, \bar{y}^n$ are solutions of

$$\left(\bar{\Gamma}^p_{jk}\frac{\partial \bar{y}^i}{\partial \bar{x}^p} - \frac{\partial^2 \bar{y}^i}{\partial \bar{x}^j\,\partial \bar{x}^k}\right)\frac{\partial \bar{x}^j}{\partial \bar{y}^\alpha}\,\frac{\partial \bar{x}^k}{\partial \bar{y}^\beta}\,\bar{y}^\alpha\,\bar{y}^\beta = 0. \tag{8.8}$$

If we substitute into this the value of $\bar{\Gamma}^p_{jk}$ from (3.7) we obtain

$$\left[\left(\Gamma^t_{qr}\frac{\partial x^q}{\partial \bar{x}^j}\,\frac{\partial x^r}{\partial \bar{x}^k}\,\frac{\partial \bar{x}^p}{\partial x^t} + \frac{\partial^2 x^t}{\partial \bar{x}^j\,\partial \bar{x}^k}\,\frac{\partial \bar{x}^p}{\partial x^t}\right)\frac{\partial \bar{y}^i}{\partial \bar{x}^p} - \frac{\partial^2 \bar{y}^i}{\partial \bar{x}^j\,\partial \bar{x}^k}\right]\frac{\partial \bar{x}^j}{\partial \bar{y}^\alpha}\,\frac{\partial \bar{x}^k}{\partial \bar{y}^\beta}\,\bar{y}^\alpha\,\bar{y}^\beta = 0,$$

or

$$\begin{aligned} &\left(\Gamma^t_{qr}\frac{\partial \bar{y}^i}{\partial x^t} - \frac{\partial^2 \bar{y}^i}{\partial x^q\,\partial x^r}\right)\frac{\partial x^q}{\partial \bar{y}^\alpha}\,\frac{\partial x^r}{\partial \bar{y}^\beta}\,\bar{y}^\alpha\,\bar{y}^\beta \\ &+\left(\frac{\partial^2 x^t}{\partial \bar{x}^j\,\partial \bar{x}^k}\,\frac{\partial \bar{y}^i}{\partial x^t} - \frac{\partial^2 \bar{y}^i}{\partial \bar{x}^j\,\partial \bar{x}^k} + \frac{\partial^2 \bar{y}^i}{\partial x^q\,\partial x^r}\,\frac{\partial x^q}{\partial \bar{x}^j}\,\frac{\partial x^r}{\partial \bar{x}^k}\right)\frac{\partial \bar{x}^j}{\partial \bar{y}^\alpha}\,\frac{\partial \bar{x}^k}{\partial \bar{y}^\beta}\,\bar{y}^\alpha\,\bar{y}^\beta = 0. \end{aligned} \tag{8.9}$$

The parenthesis in the second term is identically zero. Hence (8.9) is the same differential equation as (8.2).

By definition the normal coördinates must satisfy the initial conditions

$$\bar{y}^i = 0 \text{ when } \bar{x}^i = \bar{q}^i,$$

and

$$\delta^i_j = \frac{\partial \bar{y}^i}{\partial \bar{x}^j} \text{ when } \bar{x}^i = \bar{q}^i.$$

Hence we must have

$$\frac{\partial \bar{y}^i}{\partial x^j} = \frac{\partial \bar{x}^i}{\partial x^j} \text{ when } \bar{x}^i = \bar{q}^i.$$

The value of $\partial \bar{x}^i / \partial x^j$ when $\bar{x}^i = \bar{q}^i$ is determined by (3.1). Let us call it a^i_j. Then by the theorem regarding the formula (8.7),

$$\bar{y}^i = a^i_\alpha y^\alpha \tag{8.10}$$

is a solution of (8.9) determined by the conditions

$$\bar{y}^i = 0 \quad \text{and} \quad \frac{\partial \bar{y}^i}{\partial \bar{x}^j} = \delta^i_j \qquad \text{when } x^i = q^i.$$

Hence when the coördinates x undergo an arbitrary transformation (3.1) the normal coördinates undergo a linear transformation (8.10) the coefficients of which are given by

$$\frac{\partial f^i(q)}{\partial x^j} = a^i_j. \tag{8.11}$$

9. **The normal tensors.** Since C^i_{jk} is symmetric in j and k and ξ^i is entirely arbitrary it follows from (7.6) that C^i_{jk} vanishes at the origin of normal coördinates, i.e.,

$$(C^i_{jk})_0 = 0. \tag{9.1}$$

Hence the power series for C^i_{jk} takes the form

$$C^i_{jk} = A^i_{jk\alpha} y^\alpha + \frac{1}{2!} A^i_{jk\alpha\beta} y^\alpha y^\beta + \frac{1}{3!} A^i_{jk\alpha\beta\gamma} y^\alpha y^\beta y^\gamma + \cdots \tag{9.2}$$

in which the A's are the derivatives of C^i_{jk} evaluated at the origin, i.e.,

$$A^i_{jk\alpha\cdots\tau} = \left(\frac{\partial^n C^i_{jk}}{\partial y^\alpha \cdots \partial y^\tau} \right)_0. \tag{9.3}$$

The equation (9.3) can be taken as defining $A^i_{jk\alpha\ldots\tau}$ as a set of functions of $(x^1, x^2, \ldots, x^n)$. At any point $(p^1, p^2, \ldots, p^n)$, $A^i_{jk\alpha\ldots\tau}$ is equal to the right

hand member of (9.3) evaluated in the system of normal coördinates having $(p^1, p^2, \ldots, p^n)$ as origin. The functions so defined are tensors. For consider a transformation from x to $\bar{x}$ and the transformation which it produces from y to $\bar{y}$. By (3.7) we have

$$\bar{C}^{\alpha}_{jk} \frac{\partial y^i}{\partial \bar{y}^{\alpha}} = C^i_{\beta\gamma} \frac{\partial y^\beta}{\partial \bar{y}^j} \frac{\partial y^\gamma}{\partial \bar{y}^k} \tag{9.4}$$

and from (9.2)

$$C^i_{\beta\gamma} \frac{\partial y^\beta}{\partial \bar{y}^j} \frac{\partial y^\gamma}{\partial \bar{y}^k} = A^i_{\beta\gamma\delta} \frac{\partial y^\beta}{\partial \bar{y}^j} \frac{\partial y^\gamma}{\partial \bar{y}^k} y^\delta + \frac{1}{2!} A^i_{\beta\gamma\delta\varepsilon} \frac{\partial y^\beta}{\partial \bar{y}^j} \frac{\partial y^\gamma}{\partial \bar{y}^k} y^\delta y^\varepsilon + \cdots.$$

Hence

$$\begin{aligned} \bar{C}^{\alpha}_{jk} \frac{\partial y^i}{\partial \bar{y}^{\alpha}} = {} & A^i_{\beta\gamma\delta} \frac{\partial y^\beta}{\partial \bar{y}^j} \frac{\partial y^\gamma}{\partial \bar{y}^k} \frac{\partial y^\delta}{\partial \bar{y}^\mu} \bar{y}^\mu \\ & + \frac{1}{2!} A^i_{\beta\gamma\delta\varepsilon} \frac{\partial y^\beta}{\partial \bar{y}^j} \frac{\partial y^\gamma}{\partial \bar{y}^k} \frac{\partial y^\delta}{\partial \bar{y}^\mu} \frac{\partial y^\varepsilon}{\partial \bar{y}^\nu} \bar{y}^\mu \bar{y}^\nu + \cdots. \end{aligned}$$

Comparing this with the equation

$$\bar{C}^i_{jk} = \bar{A}^i_{jk\mu} \bar{y}^\mu + \frac{1}{2!} \bar{A}^i_{jk\mu\nu} \bar{y}^\mu \bar{y}^\nu + \cdots$$

we have

$$\bar{A}^{\alpha}_{jkl\ldots m} \frac{\partial y^i}{\partial \bar{y}^{\alpha}} = A^{\alpha}_{\beta\gamma\delta\ldots\varepsilon} \frac{\partial y^\beta}{\partial \bar{y}^j} \frac{\partial y^\gamma}{\partial \bar{y}^k} \frac{\partial y^\delta}{\partial \bar{y}^l} \cdots \frac{\partial y^\varepsilon}{\partial \bar{y}^m}.$$

If we make the substitution

$$\frac{\partial y^i}{\partial \bar{y}^j} = \left(\frac{\partial x^i}{\partial \bar{x}^j}\right)_0, \tag{9.5}$$

then

$$\bar{A}^{\alpha}_{jkl\ldots m} \frac{\partial x^i}{\partial \bar{x}^{\alpha}} = A^{\alpha}_{\beta\gamma\delta\ldots\varepsilon} \frac{\partial x^\beta}{\partial \bar{x}^j} \frac{\partial x^\gamma}{\partial \bar{x}^k} \frac{\partial x^\delta}{\partial \bar{x}^l} \cdots \frac{\partial x^\varepsilon}{\partial \bar{x}^m}, \tag{9.6}$$

where A is regarded as a function of $(x^1, x^2, \ldots, x^n)$ and $\bar{A}$ as a function of $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$. This shows that $A^i_{jkl\ldots m}$ is a tensor which is contravariant in i and covariant in $jkl \cdots m$. We shall call it a *normal tensor* because of its definition in terms of normal coördinates.

By their definition (cf. (9.3)) these tensors are symmetric in the first two subscripts and also in the remaining ones, i. e.,

$$A^i_{jk\alpha\beta\ldots\tau} = A^i_{kj\alpha\beta\ldots\tau}, \tag{9.7}$$

$$A^i_{jk\alpha\beta\ldots\tau} = A^i_{jk\gamma\delta\ldots\sigma}, \tag{9.8}$$

where $\gamma\delta\ldots\sigma$ is intended to represent any permutation of $\alpha\beta\ldots\tau$.

If we multiply (9.2) by $y^j y^k$ and sum, the left member is zero by (7.7) and the right member is a multiple power series the coefficient of each term of which must be zero. It therefore follows that

$$A^i_{jk\alpha} + A^i_{k\alpha j} + A^i_{\alpha jk} = 0, \tag{9.9}$$

$$A^i_{jk\alpha\beta} + A^i_{j\alpha k\beta} + A^i_{j\beta\alpha k} + A^i_{k\alpha j\beta} + A^i_{k\beta j\alpha} + A^i_{\alpha\beta jk} = 0. \tag{9.10}$$

and in general

$$S(A^i_{jk\alpha\beta\ldots\tau}) = 0, \tag{9.11}$$

where $S(\ )$ stands for the sum of the $N(N-1)/2$ terms obtainable from the one in the parenthesis and not identical because of (9.7) and (9.8).

The tensors A are expressible in terms of the functions Γ^i_{jk} and their derivatives. If we differentiate (7.5) we obtain

$$\begin{aligned} \frac{\partial C^\alpha_{jk}}{\partial y^l}\frac{\partial x^i}{\partial y^\alpha} + C^\alpha_{jk}\frac{\partial^2 x^i}{\partial y^\alpha \partial y^l} &= \frac{\partial^3 x^i}{\partial y^j \partial y^k \partial y^l} \\ &+ \frac{\partial \Gamma^i_{\beta\gamma}}{\partial x^\delta}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x^\gamma}{\partial y^k}\frac{\partial x^\delta}{\partial y^l} + \Gamma^i_{\beta\gamma}\frac{\partial^2 x^\beta}{\partial y^j \partial y^l}\frac{\partial x^\gamma}{\partial y^k} + \Gamma^i_{\beta\gamma}\frac{\partial x^\beta}{\partial y^j}\frac{\partial^2 x^\gamma}{\partial y^k \partial y^l}. \end{aligned} \tag{9.12}$$

Substituting the values of the partial derivatives of x with regard to the y's as computed from (6.6) or (8.5) for the origin of normal coördinates, we find

$$A^i_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \Gamma^i_{jkl} - \Gamma^i_{\beta k}\Gamma^\beta_{jl} - \Gamma^i_{j\beta}\Gamma^\beta_{kl}. \tag{9.13}$$

If we differentiate (9.12) again we obtain

$$
\begin{aligned}
A^i_{jklm} = {} & \frac{\partial^2 \Gamma^i_{jk}}{\partial x^l \partial x^m} - \Gamma^i_{jklm} - \frac{\partial \Gamma^i_{j\beta}}{\partial x^l} \Gamma^\beta_{km} - \frac{\partial \Gamma^i_{\beta k}}{\partial x^l} \Gamma^\beta_{jm} - \frac{\partial \Gamma^i_{j\beta}}{\partial x^m} \Gamma^\beta_{kl} \\
& - \frac{\partial \Gamma^i_{\beta k}}{\partial x^m} \Gamma^\beta_{jl} - \frac{\partial \Gamma^i_{jk}}{\partial x^\beta} \Gamma^\beta_{lm} - \Gamma^i_{j\beta} \Gamma^\beta_{klm} - \Gamma^i_{\beta k} \Gamma^\beta_{jlm} \qquad (9.14) \\
& + A^\beta_{jkl} \Gamma^i_{\beta m} + A^\beta_{jkm} \Gamma^i_{\beta l} + \Gamma^i_{\beta\gamma} \Gamma^\beta_{jl} \Gamma^\gamma_{km} + \Gamma^i_{\beta\gamma} \Gamma^\beta_{jm} \Gamma^\gamma_{kl}.
\end{aligned}
$$

It is evident that a continuation of this process will determine the explicit formulas for any number of the A's.

10. **Covariant differentiation.** Covariant differentiation is a process by which from a given tensor there may be formed a new tensor with one more covariant index. Let $T^{lm\ldots n}_{ij\ldots k}$ be any tensor referred to arbitrary coördinates $(x^1, x^2, \ldots, x^n)$ which is contravariant in $(l, m, \ldots, n)$ and covariant in $(i, j, \ldots, k)$. Let $t^{lm\ldots n}_{ij\ldots k}$ be the components of $T^{lm\ldots n}_{ij\ldots k}$ in the normal coördinate system $(y^1, y^2, \ldots, y^n)$ which is determined by the x-coördinate system at the point $(q^1, q^2, \ldots, q^n)$. The equation

$$
T^{lm\ldots n}_{ij\ldots k,p} = \left(\frac{\partial t^{lm\ldots n}_{ij\ldots k}}{\partial y^p} \right)_0 \qquad (10.1)
$$

defines a set of functions of x which turn out to be the components of a tensor. In the Riemann geometry this tensor is the same as the covariant derivative of T according to the definition of Ricci and Levi-Civita. Hence we shall call it by the same name in the general case. The subscripts arising by covariant differentiation will be separated from those originally present in the tensor by a comma.

Let us now prove that the functions $T^{lm\ldots n}_{ij\ldots k,p}$ actually are the components of a tensor. Let the functions T and t become $\overline{T}$ and $\bar{t}$ respectively under the arbitrary transformation (3.1). This gives the equations

$$
\overline{T}^{lm\ldots n}_{ij\ldots k} = T^{\delta\varepsilon\ldots\nu}_{\alpha\beta\ldots\gamma} \frac{\partial \bar{x}^l}{\partial x^\delta} \frac{\partial \bar{x}^m}{\partial x^\varepsilon} \cdots \frac{\partial \bar{x}^n}{\partial x^\nu} \frac{\partial x^\alpha}{\partial \bar{x}^i} \frac{\partial x^\beta}{\partial \bar{x}^j} \cdots \frac{\partial x^\gamma}{\partial \bar{x}^k}, \qquad (10.2)
$$

$$
\bar{t}^{lm\ldots n}_{ij\ldots k} = t^{\delta\varepsilon\ldots\nu}_{\alpha\beta\ldots\gamma} \frac{\partial \bar{y}^l}{\partial y^\delta} \frac{\partial \bar{y}^m}{\partial y^\varepsilon} \cdots \frac{\partial \bar{y}^n}{\partial y^\nu} \frac{\partial y^\alpha}{\partial \bar{y}^i} \frac{\partial y^\beta}{\partial \bar{y}^j} \cdots \frac{\partial y^\gamma}{\partial \bar{y}^k}. \qquad (10.3)
$$

Since the derivatives in (10.3) are constants we obtain by partial differentiation

$$(10.4)\quad \frac{\partial \bar{t}^{lm\ldots n}_{ij\ldots k}}{\partial \bar{y}^p} = \frac{\partial t^{\delta\varepsilon\ldots\nu}_{\alpha\beta\ldots\gamma}}{\partial y^\omega}\frac{\partial \bar{y}^l}{\partial y^\delta}\frac{\partial \bar{y}^m}{\partial y^\varepsilon}\cdots\frac{\partial \bar{y}^n}{\partial y^\nu}\frac{\partial y^\alpha}{\partial \bar{y}^i}\frac{\partial y^\beta}{\partial \bar{y}^j}\cdots\frac{\partial y^\gamma}{\partial \bar{y}^k}\frac{\partial y^\omega}{\partial \bar{y}^p},$$

and, hence, at the point q we have

$$(10.5)\quad \bar{T}^{lm\ldots n}_{ij\ldots k,p} = T^{\delta\varepsilon\ldots\nu}_{\alpha\beta\ldots\gamma,\omega}\frac{\partial \bar{x}^l}{\partial x^\delta}\frac{\partial \bar{x}^m}{\partial x^\varepsilon}\cdots\frac{\partial \bar{x}^n}{\partial x^\nu}\frac{\partial x^\alpha}{\partial \bar{x}^i}\frac{\partial x^\beta}{\partial \bar{x}^j}\cdots\frac{\partial x^\gamma}{\partial \bar{x}^k}\frac{\partial x^\omega}{\partial \bar{x}^p}.$$

Since the point q is arbitrary, $T^{lm\ldots n}_{ij\ldots k,p}$ is a tensor which is contravariant in $(l, m, \ldots, n)$ and covariant in $(i, j, \ldots, k, p)$.

Let us next evaluate $T^{lm\ldots n}_{ij\ldots k,p}$ in terms of the Γ's and the original tensor $T^{lm\ldots n}_{ij\ldots k}$. To do this we differentiate the equations

$$(10.6)\quad t^{lm\ldots n}_{ij\ldots k} = T^{\delta\varepsilon\ldots\nu}_{\alpha\beta\ldots\gamma}\frac{\partial y^l}{\partial x^\delta}\frac{\partial y^m}{\partial x^\varepsilon}\cdots\frac{\partial y^n}{\partial x^\nu}\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\cdots\frac{\partial x^\gamma}{\partial y^k},$$

obtaining

$$(10.7)\quad \begin{aligned}\frac{\partial t^{lm\ldots n}_{ij\ldots k}}{\partial y^p} &= \frac{\partial T^{\delta\ldots\nu}_{\alpha\ldots\gamma}}{\partial x^\sigma}\frac{\partial y^l}{\partial x^\delta}\cdots\frac{\partial y^n}{\partial x^\nu}\frac{\partial x^\alpha}{\partial y^i}\cdots\frac{\partial x^\gamma}{\partial y^k}\frac{\partial x^\sigma}{\partial y^p}\\ &\quad + T^{\delta\ldots\nu}_{\alpha\ldots\gamma}\frac{\partial^2 y^l}{\partial x^\delta\,\partial x^\sigma}\frac{\partial x^\sigma}{\partial y^p}\cdots\frac{\partial y^n}{\partial x^\nu}\frac{\partial x^\alpha}{\partial y^i}\cdots\frac{\partial x^\gamma}{\partial y^k}+\cdots\\ &\quad + T^{\delta\ldots\nu}_{\alpha\ldots\gamma}\frac{\partial y^l}{\partial x^\delta}\cdots\frac{\partial y^n}{\partial x^\nu}\frac{\partial x^\alpha}{\partial y^i}\cdots\frac{\partial^2 x^\gamma}{\partial y^k\,\partial y^p}.\end{aligned}$$

At the origin of normal coördinates

$$(10.8)\quad \frac{\partial x^i}{\partial y^j} = \frac{\partial y^i}{\partial x^j} = \delta^i_j, \qquad \frac{\partial^2 y^i}{\partial x^j\,\partial x^k} = -\frac{\partial^2 x^i}{\partial y^j\,\partial y^k} = (\Gamma^i_{jk})_0,$$

as follows directly from (8.5) and its inverse (8.6). The substitution of (10.8) into (10.7) then yields

$$(10.9)\quad \begin{aligned}T^{lm\ldots n}_{ij\ldots k,p} &= \frac{\partial T^{lm\ldots n}_{ij\ldots k}}{\partial x^p} + \Gamma^l_{\alpha p}T^{\alpha m\ldots n}_{ij\ldots k} + \Gamma^m_{\alpha p}T^{l\alpha\ldots n}_{ij\ldots k}+\cdots\\ &\quad + \Gamma^n_{\alpha p}T^{lm\ldots\alpha}_{ij\ldots k} - \Gamma^\alpha_{ip}T^{lm\ldots n}_{\alpha j\ldots k} - \Gamma^\alpha_{jp}T^{lm\ldots n}_{i\alpha\ldots k} - \cdots - \Gamma^\alpha_{kp}T^{lm\ldots n}_{ij\ldots\alpha}.\end{aligned}$$

By using the tensors

$$D^{\sigma ab\dots n}_{\alpha\tau\beta\dots\nu} = \delta^{\sigma}_{\alpha}\,\delta^{a}_{\tau}\,\delta^{b}_{\beta}\cdots\delta^{n}_{\nu} + \delta^{\sigma}_{\beta}\,\delta^{a}_{\alpha}\,\delta^{b}_{\tau}\cdots\delta^{n}_{\nu} + \cdots + \delta^{\sigma}_{\nu}\,\delta^{a}_{\alpha}\,\delta^{b}_{\beta}\cdots\delta^{n}_{\tau} \tag{10.10}$$

and

$$E^{\alpha\tau\beta\dots\nu}_{\sigma ab\dots n} = \delta^{\alpha}_{\sigma}\,\delta^{\tau}_{a}\,\delta^{\beta}_{b}\cdots\delta^{\nu}_{n} + \delta^{\beta}_{\sigma}\,\delta^{\alpha}_{a}\,\delta^{\tau}_{b}\cdots\delta^{\nu}_{n} + \cdots + \delta^{\nu}_{\sigma}\,\delta^{\alpha}_{a}\,\delta^{\beta}_{b}\cdots\delta^{\tau}_{n}, \tag{10.11}$$

the formula (10.9) may be written in the form

$$\begin{aligned} T^{lm\dots n}_{ij\dots k,p} = \frac{\partial T^{lm\dots n}_{ij\dots k}}{\partial x^{p}} &+ \Gamma^{\tau}_{\sigma p}\, T^{\alpha\beta\dots\gamma}_{ij\dots k}\, D^{\sigma lm\dots n}_{\alpha\tau\beta\dots\gamma} \\ &- \Gamma^{\sigma}_{\tau p}\, T^{lm\dots n}_{\alpha\beta\dots\gamma}\, E^{\alpha\tau\beta\dots\gamma}_{\sigma ij\dots k}. \end{aligned} \tag{10.12}$$

The covariant derivatives of the sum and of the product of two tensors with the same number of covariant and contravariant indices are formed by the same rules as hold in the differential calculus. That is, if

$$T^{lm\dots n}_{ij\dots k} = A^{lm\dots n}_{ij\dots k} + B^{lm\dots n}_{ij\dots k} \tag{10.13}$$

then

$$T^{lm\dots n}_{ij\dots k,p} = A^{lm\dots n}_{ij\dots k,p} + B^{lm\dots n}_{ij\dots k,p}, \tag{10.14}$$

and if

$$T^{lm\dots n}_{ij\dots k} = A^{l\dots u}_{i\dots s}\cdot B^{v\dots n}_{t\dots k} \tag{10.15}$$

then

$$T^{lm\dots n}_{ij\dots k,p} = A^{l\dots u}_{i\dots s}\cdot B^{v\dots n}_{t\dots k,p} + A^{l\dots u}_{i\dots s,p}\cdot B^{v\dots n}_{t\dots k}. \tag{10.16}$$

These formulas follow without difficulty from (10.1).

11. A generalization of covariant differentiation. By repeated differentiation of (10.3) we obtain

$$\begin{aligned} \frac{\partial^{r}\,\bar t^{lm\dots n}_{ij\dots k}}{\partial \bar y^{p}\dots\partial \bar y^{q}} = \frac{\partial^{r}\, t^{\delta\varepsilon\dots\nu}_{\alpha\beta\dots\gamma}}{\partial y^{\sigma}\dots\partial y^{\tau}}\,\frac{\partial \bar y^{l}}{\partial y^{\delta}}\,\frac{\partial \bar y^{m}}{\partial y^{\varepsilon}}\cdots \\ \cdots\frac{\partial \bar y^{n}}{\partial y^{\nu}}\,\frac{\partial y^{\alpha}}{\partial \bar y^{i}}\,\frac{\partial y^{\beta}}{\partial \bar y^{j}}\cdots\frac{\partial y^{\gamma}}{\partial \bar y^{k}}\,\frac{\partial y^{\sigma}}{\partial \bar y^{p}}\cdots\frac{\partial y^{\tau}}{\partial \bar y^{q}}. \end{aligned} \tag{11.1}$$

This shows that a set of tensors $T^{lm\ldots n}_{ij\ldots k,p\ldots q}$ are defined by

$$(11.2)\qquad T^{lm\ldots n}_{ij\ldots k,p\ldots q} = \left(\frac{\partial^r\, t^{lm\ldots n}_{ij\ldots k}}{\partial y^p \ldots \partial y^q}\right)_0 \qquad (r = 1, 2, 3, \ldots),$$

where the derivatives on the right are evaluated at the origin of normal coördinates. For $r = 1$, $T^{lm\ldots n}_{ij\ldots k,p\ldots q}$ is the ordinary covariant derivative that we have just considered. The tensors $T^{lm\ldots n}_{ij\ldots k,p\ldots q}$ form a group of tensors that may be derived from a given tensor. We shall refer to the general tensor of this group, namely, $T^{lm\ldots n}_{ij\ldots k,p\ldots q}$, as the rth extension of $T^{lm\ldots n}_{ij\ldots k}$, r being the number of indices $p, \ldots, q$. By its definition this tensor is symmetric with respect to the indices $p, \ldots, q$. The operation of forming the extension of a tensor may be repeated any number of times. For example, $T_{ij,pq,r,stu}$ is the third extension of the first extension of the second extension of T_{ij}.

The rth extension of the sum of two tensors which are of the same order in their covariant and contravariant indices is equal to the sum of the rth extensions of the two tensors, i. e.,

$$(11.3)\qquad (A+B)^{lm\ldots n}_{ij\ldots k,p\ldots q} = A^{lm\ldots n}_{ij\ldots k,p\ldots q} + B^{lm\ldots n}_{ij\ldots k,p\ldots q}.$$

This follows directly from the character of the tensor transformation. The formula for the covariant derivative (first extension) of the product of two tensors does not apply, however, for the case of the rth extension $(r > 1)$. For let the tensor T be equal to the product of two tensors as in the equation (10.15). If T, A, B become t, a, b in a normal coördinate system $(y^1, y^2, \ldots, y^n)$ we have in this system

$$(11.4)\qquad t^{lm\ldots n}_{ij\ldots k} = a^{l\ldots u}_{i\ldots s} \cdot b^{v\ldots n}_{t\ldots k}.$$

The formula for the rth extension of T is obtained by carrying out the differentiation indicated in the following equation:

$$(11.5)\qquad \frac{\partial^r\, t^{lm\ldots n}_{ij\ldots k}}{\partial y^p \ldots \partial y^q} = \frac{\partial^r}{\partial y^p \ldots \partial y^q}\left(a^{l\ldots u}_{i\ldots s} \cdot b^{v\ldots n}_{t\ldots k}\right).$$

This formula has 2^r terms.

Any tensor $T^{lm\ldots n}_{ij\ldots k,p\ldots q}$ may be expressed in terms of the Γ's and the original tensor $T^{lm\ldots n}_{ij\ldots k}$ by the same process that we have used for the case

of the tensor $T^{lm\ldots n}_{ij\ldots k,p}$. That is we have to take the successive partial derivatives of both members of (10.6) and substitute in these equations the equations (10.8) and

$$\frac{\partial^3 x^i}{\partial y^j\,\partial y^k\,\partial y^l} = -(\Gamma^i_{jkl})_0, \qquad \frac{\partial^3 y^i}{\partial x^j\,\partial x^k\,\partial x^l} = (A^i_{jkl})_0, \tag{11.6}$$

and so on. It is evident that formulas for extensions of all kinds can be obtained by this process. Instead of giving the general formulas, however, we shall set down the first extensions of the first four kinds for the covariant vector and the covariant tensor of the second order. In these formulas S is used to indicate the sum of all distinct terms which can be formed from the one in the parenthesis by replacing the given combination of the subscripts p, q or p, q, r or p, q, r, s by arbitrary combinations of these subscripts. Thus

$$S\left(\frac{\partial^2 T_i}{\partial x^\alpha\,\partial x^p}\,\Gamma^\alpha_{qr}\right) = \frac{\partial^2 T_i}{\partial x^\alpha\,\partial x^p}\,\Gamma^\alpha_{qr} + \frac{\partial^2 T_i}{\partial x^\alpha\,\partial x^q}\,\Gamma^\alpha_{pr} + \frac{\partial^2 T_i}{\partial x^\alpha\,\partial x^r}\,\Gamma^\alpha_{pq}.$$

$$T_{i,p} = \frac{\partial T_i}{\partial x^p} - T_\alpha\,\Gamma^\alpha_{ip}; \tag{11.7}$$

$$T_{i,pq} = \frac{\partial^2 T_i}{\partial x^p\,\partial x^q} - \frac{\partial T_i}{\partial x^\alpha}\,\Gamma^\alpha_{pq} - S\left(\frac{\partial T_\alpha}{\partial x^p}\,\Gamma^\alpha_{iq}\right) - T_\alpha\,\Gamma^\alpha_{ipq}; \tag{11.8}$$

$$\begin{aligned} T_{i,pqr} = {} & \frac{\partial^3 T_i}{\partial x^p\,\partial x^q\,\partial x^r} - S\left(\frac{\partial^2 T_i}{\partial x^\alpha\,\partial x^p}\,\Gamma^\alpha_{qr}\right) - S\left(\frac{\partial^2 T_\alpha}{\partial x^p\,\partial x^q}\,\Gamma^\alpha_{ir}\right) \\ & + S\left(\frac{\partial T_\alpha}{\partial x^\beta}\,\Gamma^\alpha_{ip}\,\Gamma^\beta_{qr}\right) - S\left(\frac{\partial T_\alpha}{\partial x^p}\,\Gamma^\alpha_{iqr}\right) - \frac{\partial T_i}{\partial x^\alpha}\,\Gamma^\alpha_{pqr} - T_\alpha\,\Gamma^\alpha_{ipqr}; \end{aligned} \tag{11.9}$$

$$\begin{aligned} T_{i,pqrs} = {} & \frac{\partial^4 T_i}{\partial x^p\,\partial x^q\,\partial x^r\,\partial x^s} - S\left(\frac{\partial^3 T_i}{\partial x^\alpha\,\partial x^p\,\partial x^q}\,\Gamma^\alpha_{rs}\right) - S\left(\frac{\partial^3 T_\alpha}{\partial x^p\,\partial x^q\,\partial x^r}\,\Gamma^\alpha_{is}\right) \\ & + S\left(\frac{\partial^2 T_i}{\partial x^\alpha\,\partial x^\beta}\,\Gamma^\alpha_{pq}\,\Gamma^\beta_{rs}\right) + S\left(\frac{\partial^2 T_\alpha}{\partial x^\beta\,\partial x^p}\,\Gamma^\alpha_{iq}\,\Gamma^\beta_{rs}\right) - S\left(\frac{\partial^2 T_i}{\partial x^\alpha\,\partial x^p}\,\Gamma^\alpha_{qrs}\right) \\ & - S\left(\frac{\partial^2 T_\alpha}{\partial x^p\,\partial x^q}\,\Gamma^\alpha_{irs}\right) + S\left(\frac{\partial T_\alpha}{\partial x^\beta}\,\Gamma^\alpha_{ip}\,\Gamma^\beta_{qrs}\right) + S\left(\frac{\partial T_\alpha}{\partial x^\beta}\,\Gamma^\alpha_{ipq}\,\Gamma^\beta_{rs}\right) \\ & - S\left(\frac{\partial T_\alpha}{\partial x^p}\,\Gamma^\alpha_{iqrs}\right) - \frac{\partial T_i}{\partial x^\alpha}\,\Gamma^\alpha_{pqrs} - T_\alpha\,\Gamma^\alpha_{ipqrs}; \end{aligned} \tag{11.10}$$

$$T_{ij,p} = \frac{\partial T_{ij}}{\partial x^p} - T_{\alpha j}\Gamma^\alpha_{ip} - T_{i\alpha}\Gamma^\alpha_{jp}; \tag{11.11}$$

$$\begin{aligned} T_{ij,pq} = {} & \frac{\partial^2 T_{ij}}{\partial x^p\,\partial x^q} - \frac{\partial T_{ij}}{\partial x^\alpha}\Gamma^\alpha_{pq} - S\left(\frac{\partial T_{\alpha j}}{\partial x^p}\Gamma^\alpha_{iq}\right) - S\left(\frac{\partial T_{i\alpha}}{\partial x^p}\Gamma^\alpha_{jq}\right) \\ & + T_{\alpha\beta}\,S(\Gamma^\alpha_{ip}\Gamma^\beta_{jq}) - T_{\alpha j}\Gamma^\alpha_{ipq} - T_{i\alpha}\Gamma^\alpha_{jpq}; \end{aligned} \tag{11.12}$$

$$\begin{aligned} T_{ij,pqr} = {} & \frac{\partial^3 T_{ij}}{\partial x^p\,\partial x^q\,\partial x^r} - S\left(\frac{\partial^2 T_{\alpha j}}{\partial x^p\,\partial x^q}\Gamma^\alpha_{ir}\right) - S\left(\frac{\partial^2 T_{i\alpha}}{\partial x^p\,\partial x^q}\Gamma^\alpha_{jr}\right) \\ & - S\left(\frac{\partial^2 T_{ij}}{\partial x^\alpha\,\partial x^p}\Gamma^\alpha_{qr}\right) + \frac{\partial T_{\alpha j}}{\partial x^\beta}S(\Gamma^\alpha_{ip}\Gamma^\beta_{qr}) + \frac{\partial T_{i\alpha}}{\partial x^\beta}S(\Gamma^\alpha_{jp}\Gamma^\beta_{qr}) \\ & + S\left(\frac{\partial T_{\alpha\beta}}{\partial x^p}\Gamma^\alpha_{iq}\Gamma^\beta_{jr}\right) - S\left(\frac{\partial T_{\alpha j}}{\partial x^p}\Gamma^\alpha_{iqr}\right) - S\left(\frac{\partial T_{i\alpha}}{\partial x^p}\Gamma^\alpha_{jqr}\right) \\ & - \frac{\partial T_{ij}}{\partial x^\alpha}\Gamma^\alpha_{pqr} + T_{\alpha\beta}\,S(\Gamma^\alpha_{ip}\Gamma^\beta_{jqr}) + T_{\alpha\beta}\,S(\Gamma^\alpha_{ipq}\Gamma^\beta_{jr}) \\ & - T_{\alpha j}\Gamma^\alpha_{ipqr} - T_{i\alpha}\Gamma^\alpha_{jpqr}; \end{aligned} \tag{11.13}$$

$$\begin{aligned} T_{ij,pqrs} = {} & \frac{\partial^4 T_{ij}}{\partial x^p\,\partial x^q\,\partial x^r\,\partial x^s} - S\left(\frac{\partial^3 T_{\alpha j}}{\partial x^p\,\partial x^q\,\partial x^r}\Gamma^\alpha_{is}\right) - S\left(\frac{\partial^3 T_{i\alpha}}{\partial x^p\,\partial x^q\,\partial x^r}\Gamma^\alpha_{js}\right) \\ & - S\left(\frac{\partial^3 T_{ij}}{\partial x^\alpha\,\partial x^p\,\partial x^q}\Gamma^\alpha_{rs}\right) + S\left(\frac{\partial^2 T_{\alpha\beta}}{\partial x^p\,\partial x^q}\Gamma^\alpha_{ir}\Gamma^\beta_{js}\right) + S\left(\frac{\partial^2 T_{\alpha j}}{\partial x^\beta\,\partial x^p}\Gamma^\alpha_{iq}\Gamma^\beta_{rs}\right) \\ & + S\left(\frac{\partial^2 T_{i\alpha}}{\partial x^\beta\,\partial x^p}\Gamma^\alpha_{jq}\Gamma^\beta_{rs}\right) - S\left(\frac{\partial^2 T_{i\alpha}}{\partial x^p\,\partial x^q}\Gamma^\alpha_{jrs}\right) - S\left(\frac{\partial^2 T_{ij}}{\partial x^\alpha\,\partial x^p}\Gamma^\alpha_{qrs}\right) \\ & + S\left(\frac{\partial^2 T_{ij}}{\partial x^\alpha\,\partial x^\beta}\Gamma^\alpha_{pq}\Gamma^\beta_{rs}\right) - S\left(\frac{\partial^2 T_{\alpha j}}{\partial x^p\,\partial x^q}\Gamma^\alpha_{irs}\right) - S\left(\frac{\partial T_{\alpha j}}{\partial x^p}\Gamma^\alpha_{iqrs}\right) \\ & + S\left(\frac{\partial T_{\alpha\beta}}{\partial x^p}\Gamma^\alpha_{iqr}\Gamma^\beta_{js}\right) + S\left(\frac{\partial T_{\alpha j}}{\partial x^\beta}\Gamma^\alpha_{ipq}\Gamma^\beta_{rs}\right) + S\left(\frac{\partial T_{\alpha\beta}}{\partial x^p}\Gamma^\alpha_{iq}\Gamma^\beta_{jrs}\right) \\ & - S\left(\frac{\partial T_{\alpha\beta}}{\partial x^\gamma}\Gamma^\alpha_{ip}\Gamma^\beta_{jq}\Gamma^\gamma_{rs}\right) + S\left(\frac{\partial T_{\alpha j}}{\partial x^\beta}\Gamma^\alpha_{ip}\Gamma^\beta_{qrs}\right) - S\left(\frac{\partial T_{i\alpha}}{\partial x^p}\Gamma^\alpha_{jqrs}\right) \\ & + S\left(\frac{\partial T_{i\alpha}}{\partial x^\beta}\Gamma^\alpha_{jpq}\Gamma^\beta_{rs}\right) + S\left(\frac{\partial T_{i\alpha}}{\partial x^\beta}\Gamma^\alpha_{jp}\Gamma^\beta_{qrs}\right) - \frac{\partial T_{ij}}{\partial x^\alpha}\Gamma^\alpha_{pqrs} \\ & - T_{\alpha j}\Gamma^\alpha_{ipqrs} + T_{\alpha\beta}\,S(\Gamma^\alpha_{ipqr}\Gamma^\beta_{js}) + S(T_{\alpha\beta}\Gamma^\alpha_{ipq}\Gamma^\beta_{jrs}) \\ & + T_{\alpha\beta}\,S(\Gamma^\alpha_{ip}\Gamma^\beta_{jqrs}) - T_{i\alpha}\Gamma^\alpha_{jpqrs}. \end{aligned} \tag{11.14}$$

12. Formulas for repeated covariant differentiation. In this section we write down a few special formulas relating the tensors obtained by successive covariant differentiation to the higher extensions and the normal tensors. In each case the formula is obtained by computing the covariant derivative in question according to the formulas of §10 and evaluating at the origin of normal coördinates:

$$T_{i,p,q} = T_{i,pq} - T_\alpha A^\alpha_{ipq}; \tag{12.1}$$

$$\begin{aligned} T_{i,p,q,r} = {} & T_{i,pqr} - T_{i,\alpha} A^\alpha_{pqr} - T_{\alpha,p} A^\alpha_{iqr} - T_{\alpha,q} A^\alpha_{ipr} \\ & - T_{\alpha,r} A^\alpha_{ipq} - T_\alpha A^\alpha_{ipqr}; \end{aligned} \tag{12.2}$$

$$\begin{aligned} T_{i,p,q,r,s} = {} & T_{i,pqrs} - T_{\alpha,pq} A^\alpha_{irs} - T_{\alpha,pr} A^\alpha_{iqs} - T_{\alpha,ps} A^\alpha_{iqr} \\ & - T_{\alpha,qr} A^\alpha_{ips} - T_{\alpha,qs} A^\alpha_{ipr} - T_{\alpha,rs} A^\alpha_{ipq} - T_{i,\alpha p} A^\alpha_{qrs} \\ & - T_{i,\alpha q} A^\alpha_{prs} - T_{i,\alpha r} A^\alpha_{pqs} - T_{i,\alpha s} A^\alpha_{pqr} - T_{\alpha,p} A^\alpha_{iqrs} \\ & - T_{\alpha,q} A^\alpha_{iprs} - T_{\alpha,r} A^\alpha_{ipqs} - T_{\alpha,s} A^\alpha_{ipqr} - T_{i,\alpha} A^\alpha_{pqrs} \\ & - T_\alpha (A^\alpha_{ipqrs} - A^\alpha_{\beta pr} A^\beta_{iqs} - A^\alpha_{\beta ps} A^\beta_{iqr} - A^\alpha_{\beta ir} A^\beta_{pqs} \\ & - A^\alpha_{\beta is} A^\beta_{pqr} - A^\alpha_{\beta pq} A^\beta_{irs} - A^\alpha_{\beta iq} A^\beta_{prs} - A^\alpha_{ip\beta} A^\beta_{qrs}); \end{aligned} \tag{12.3}$$

$$T_{ij,p,q} = T_{ij,pq} - T_{\alpha j} A^\alpha_{ipq} - T_{i\alpha} A^\alpha_{jpq}; \tag{12.4}$$

$$\begin{aligned} T_{ij,p,q,r} = {} & T_{ij,pqr} - T_{\alpha j,p} A^\alpha_{iqr} - T_{\alpha j,q} A^\alpha_{ipr} - T_{\alpha j,r} A^\alpha_{ipq} \\ & - T_{i\alpha,p} A^\alpha_{jqr} - T_{i\alpha,q} A^\alpha_{jpr} - T_{i\alpha,r} A^\alpha_{jpq} - T_{ij,\alpha} A^\alpha_{pqr} \\ & - T_{\alpha j} A^\alpha_{ipqr} - T_{i\alpha} A^\alpha_{jpqr}; \end{aligned} \tag{12.5}$$

$$\begin{aligned} T_{ij,p,q,r,s} = {} & T_{ij,pqrs} - T_{\alpha j} A^\alpha_{ipqrs} - T_{i\alpha} A^\alpha_{jpqrs} \\ & - T_{\alpha j,q} A^\alpha_{iprs} - T_{\alpha j,r} A^\alpha_{ipqs} - T_{\alpha j,s} A^\alpha_{ipqr} \\ & - T_{i\alpha,q} A^\alpha_{jprs} - T_{i\alpha,r} A^\alpha_{jpqs} - T_{i\alpha,s} A^\alpha_{jpqr} \end{aligned} \tag{12.6}$$

$$
\begin{aligned}
&-T_{\alpha j,p}\,A^{\alpha}_{iqrs}-T_{i\alpha,p}\,A^{\alpha}_{jqrs}-T_{ij,\alpha}\,A^{\alpha}_{pqrs}\\
&-T_{\alpha j,qr}\,A^{\alpha}_{ips}-T_{\alpha j,qs}\,A^{\alpha}_{ipr}-T_{\alpha j,rs}\,A^{\alpha}_{ipq}\\
&-T_{i\alpha,qr}\,A^{\alpha}_{jps}-T_{i\alpha,qs}\,A^{\alpha}_{jpr}-T_{i\alpha,rs}\,A^{\alpha}_{jpq}\\
&-T_{\alpha j,p,q}\,A^{\alpha}_{irs}-T_{\alpha j,p,r}\,A^{\alpha}_{iqs}-T_{\alpha j,p,s}\,A^{\alpha}_{iqr}\\
&-T_{i\alpha,p,q}\,A^{\alpha}_{jrs}-T_{i\alpha,p,r}\,A^{\alpha}_{jqs}-T_{i\alpha,p,s}\,A^{\alpha}_{jqr}\\
&-T_{ij,\alpha,q}\,A^{\alpha}_{prs}-T_{ij,\alpha,r}\,A^{\alpha}_{pqs}-T_{ij,\alpha,s}\,A^{\alpha}_{pqr}\\
&-T_{ij,p,\alpha}\,A^{\alpha}_{qrs};
\end{aligned}
$$

$$
\begin{aligned}
T^{lm\ldots n}_{ij\ldots k,p,q} = T^{lm\ldots n}_{ij\ldots k,pq} + T^{\alpha m\ldots n}_{ij\ldots k}\,A^{l}_{\alpha pq} + \cdots + T^{lm\ldots\alpha}_{ij\ldots k}\,A^{n}_{\alpha pq}\\
- T^{lm\ldots n}_{\alpha j\ldots k}\,A^{\alpha}_{ipq} + \cdots + T^{lm\ldots n}_{ij\ldots\alpha}\,A^{\alpha}_{kpq}.
\end{aligned}
\tag{12.7}
$$

13. **A generalization of the normal tensors.** If we transform the equations (6.3) to a system of normal coördinates and make use of (7.1) we obtain the following sequence:

$$
\begin{aligned}
C^{i}_{\alpha\beta}\,\xi^{\alpha}\,\xi^{\beta} &= 0,\\
C^{i}_{\alpha\beta\gamma}\,\xi^{\alpha}\,\xi^{\beta}\,\xi^{\gamma} &= 0,\\
&\cdots\cdots\\
&\cdots\cdots\\
&\cdots\cdots\\
C^{i}_{\alpha\beta\gamma\ldots\sigma}\,\xi^{\alpha}\,\xi^{\beta}\,\xi^{\gamma}\cdots\xi^{\sigma} &= 0,\\
&\cdots\cdots\\
&\cdots\cdots\\
&\cdots\cdots
\end{aligned}
\tag{13.1}
$$

where the C's denote the corresponding functions Γ in normal coördinates. $C^{i}_{\alpha\beta\gamma\ldots\sigma}$ is symmetric in the indices $\alpha\beta\gamma\cdots\sigma$. The functions $C^{i}_{\alpha\beta}$ are related

to the Γ's by (7.5) and there are similar equations of transformation for the other C's. Thus

$$C^{\nu}_{\alpha\beta\gamma}\frac{\partial x^i}{\partial y^{\nu}} = \frac{\partial^3 x^i}{\partial y^{\alpha}\,\partial y^{\beta}\,\partial y^{\gamma}} + \Gamma^i_{\mu\nu\tau}\frac{\partial x^{\mu}}{\partial y^{\alpha}}\frac{\partial x^{\nu}}{\partial y^{\beta}}\frac{\partial x^{\tau}}{\partial y^{\gamma}} \tag{13.2}$$
$$- \frac{\partial^2 x^i}{\partial y^{\mu}\,\partial y^{\alpha}}C^{\mu}_{\beta\gamma} - \frac{\partial^2 x^i}{\partial y^{\mu}\,\partial y^{\beta}}C^{\mu}_{\gamma\alpha} - \frac{\partial^2 x^i}{\partial y^{\mu}\,\partial y^{\gamma}}C^{\mu}_{\alpha\beta}.$$

Since the ξ's are entirely arbitrary at the origin of normal coördinates it follows from (13.1) that

$$(C^i_{\alpha\beta\gamma\ldots\sigma})_0 = 0, \tag{13.3}$$

where the left member denotes the value of C at the origin of the normal coördinate system.

We may define a set of functions $A^i_{(\alpha\beta\gamma\ldots\sigma)p\ldots q}$ of $(x^1, x^2, \ldots, x^n)$ corresponding to the normal tensor $A^i_{\alpha\beta p\ldots q}$ defined by (9.3) by the equations

$$A^i_{(\alpha\beta\gamma\ldots\sigma)p\ldots q} = \left(\frac{\partial^n C^i_{\alpha\beta\gamma\ldots\sigma}}{\partial y^p\cdots\partial y^q}\right)_0, \tag{13.4}$$

in which the derivative on the right is evaluated at the origin of normal coördinates. By a method similar to that employed in § 9 we can show that $A^i_{(\alpha\beta\gamma\ldots\sigma)p\ldots q}$ possesses a tensor character, but this fact may also be inferred by observing that $A^i_{(\alpha\beta\gamma\ldots\sigma)p\ldots q}$ is expressible in terms of the normal tensors. The tensors $A^i_{(\alpha\beta\gamma\ldots\sigma)p\ldots q}$ thus constitute a generalization of the normal tensors. They are symmetric in the indices $\alpha\beta\gamma\cdots\sigma$ and $p\cdots q$. In case $A^i_{(\alpha\beta\gamma\ldots\sigma)p\ldots q}$ contains only two terms in the parenthesis it is a normal tensor and we shall then omit the parenthesis for simplicity.

The following equations express a few particular cases of the relations between the tensors $A^i_{(\alpha\beta\gamma\ldots\sigma)p\ldots q}$ and the normal tensors. These equations are obtained by differentiating the identities (6.5) referred to normal coördinates and evaluating at the origin. The symbols P and S have their previous significance (cf. § 6 and § 11) except that P operates only on the letters $\alpha\beta\gamma\delta\epsilon$ and S only on the letters p, q, r:

$$A^i_{(\alpha\beta\gamma)p} = \frac{1}{3}P(A^i_{\alpha\beta\gamma p}); \tag{13.5}$$

$$(13.6)\quad A^i_{(\alpha\beta\gamma)pq} = \frac{1}{3} P\,[A^i_{\alpha\beta\gamma pq} - 2\,S(A^i_{\nu\alpha p}\,A^\nu_{\beta\gamma q})];$$

$$(13.7)\quad A^i_{(\alpha\beta\gamma)pqr} = \frac{1}{3} P\,[A^i_{\alpha\beta\gamma pqr} - 2\,S(A^i_{\nu\alpha pq}\,A^\nu_{\beta\gamma r}) - 2\,S(A^i_{\nu\alpha p}\,A^\nu_{\beta\gamma qr})];$$

$$(13.8)\quad A^i_{(\alpha\beta\gamma\delta)p} = \frac{1}{4} P\,(A^i_{(\alpha\beta\gamma)\delta p});$$

$$(13.9)\quad A^i_{(\alpha\beta\gamma\delta)pq} = \frac{1}{4} P\,[A^i_{(\alpha\beta\gamma)\delta pq} - 3\,S(A^i_{(\nu\alpha\beta)p}\,A^\nu_{\gamma\delta q})];$$

$$(13.10)\quad A^i_{(\alpha\beta\gamma\delta\varepsilon)p} = \frac{1}{5} P\,(A^i_{\alpha\beta\gamma\delta\varepsilon p}).$$

By differentiating the equations of the type (13.2) and evaluating at the origin of normal coördinates we may express these tensors in terms of the functions Γ. For example

$$(13.11)\quad \begin{aligned} A^i_{(\alpha\beta\gamma)p} = {} & -\Gamma^i_{\alpha\beta\gamma p} + \frac{\partial \Gamma^i_{\alpha\beta\gamma}}{\partial x^p} - \Gamma^i_{\mu\alpha\beta}\,\Gamma^\mu_{\gamma p} - \Gamma^i_{\mu\alpha\gamma}\,\Gamma^\mu_{\beta p} - \Gamma^i_{\mu\beta\gamma}\,\Gamma^\mu_{\alpha p} \\ & + \Gamma^i_{\mu\alpha}\,A^\mu_{\beta\gamma p} + \Gamma^i_{\mu\beta}\,A^\mu_{\gamma\alpha p} + \Gamma^i_{\mu\gamma}\,A^\mu_{\alpha\beta p}. \end{aligned}$$

The generalized normal tensors appear in some of the formulas of extension which generalize the formulas of § 12. We here write down only the following four particular cases:

$$(13.12)\quad T_{i,p,qr} = T_{i,pqr} - T_{\alpha,q}\,A^\alpha_{ipr} - T_{\alpha,r}\,A^\alpha_{ipq} - T_\alpha\,A^\alpha_{ipqr};$$

$$(13.13)\quad T_{i,pq,r} = T_{i,pqr} - T_{i,\alpha}\,A^\alpha_{pqr} - T_{\alpha,p}\,A^\alpha_{iqr} - T_{\alpha,q}\,A^\alpha_{ipr} - T_\alpha\,A^\alpha_{(ipq)r};$$

$$(13.14)\quad \begin{aligned} T_{ij,p,qr} = {} & T_{ij,pqr} - T_{\alpha j,q}\,A^\alpha_{ipr} - T_{\alpha j,r}\,A^\alpha_{ipq} - T_{i\alpha,q}\,A^\alpha_{jpr} - T_{i\alpha,r}\,A^\alpha_{jpq} \\ & - T_{\alpha j}\,A^\alpha_{ipqr} - T_{i\alpha}\,A^\alpha_{jpqr}; \end{aligned}$$

$$(13.15)\quad \begin{aligned} T_{ij,pq,r} = {} & T_{ij,pqr} - T_{ij,\alpha}\,A^\alpha_{pqr} - T_{\alpha j,p}\,A^\alpha_{iqr} - T_{\alpha j,q}\,A^\alpha_{ipr} - T_{i\alpha,p}\,A^\alpha_{jqr} \\ & - T_{i\alpha,q}\,A^\alpha_{jpr} - T_{\alpha j}\,A^\alpha_{(ipq)r} - T_{i\alpha}\,A^\alpha_{(jpq)r}. \end{aligned}$$

The generalized normal tensors satisfy the identity

$$S(A^i_{(\alpha\beta\gamma\dots\sigma)p\dots q}) = 0, \tag{13.16}$$

where $S(\quad)$ denotes the sum of the terms obtainable from the one in the parenthesis which are not identical because of the symmetric properties of $A^i_{(\alpha\beta\gamma\dots\sigma)p\dots q}$. This identity may easily be proved by the method used for the corresponding theorem about the normal tensors in § 9.

14. **The curvature tensor**. The normal tensor A^i_{jkl} is related to the curvature tensor by the equation

$$B^i_{jkl} = A^i_{jkl} - A^i_{jlk} \tag{14.1}$$

which is immediately evident on comparing (2.7) with (9.13). The tensor character of B follows from that of A. From the definition it follows that

$$B^i_{jkl} = -B^i_{jlk}. \tag{14.2}$$

From (9.9) it follows that

$$B^i_{jkl} + B^i_{klj} + B^i_{ljk} = 0. \tag{14.3}$$

Also by solving the equations (14.1) and (9.9) for the A's we obtain

$$A^i_{jkl} = \frac{1}{3}(2\,B^i_{jkl} + B^i_{ljk}) \tag{14.4}$$

or

$$A^i_{jkl} = \frac{1}{3}(B^i_{jkl} + B^i_{kjl}). \tag{14.5}$$

If we write (9.13) in normal coördinates, differentiate, and evaluate at the origin we obtain

$$A^i_{jkl,m} = A^i_{jklm} - A^i_{(jkl)m}. \tag{14.6}$$

From this and (14.1) it follows that

$$B^i_{jkl,m} = A^i_{jklm} - A^i_{jlkm}. \tag{14.7}$$

The equations (14.7) and (9.10) may be solved for the A's giving

$$(14.8)\quad A^i_{jklm} = \frac{1}{6}(5\,B^i_{kjl,m} + 4\,B^i_{lkj,m} + 3\,B^i_{ljm,k} + 2\,B^i_{mlk,j} + B^i_{mkj,l}).$$

If in (14.7) we permute the indices k, l, m cyclically and add the three resulting equations we obtain the identity of Bianchi,

$$(14.9)\qquad B^\alpha_{jkl,m} + B^\alpha_{jlm,k} + B^\alpha_{jmk,l} = 0.$$

From (14.7) there also follows the identity,

$$(14.10)\qquad B^\alpha_{ijk,l} + B^\alpha_{kil,j} + B^\alpha_{lkj,i} + B^\alpha_{jli,k} = 0.$$

From (12.7) there follows the important identity of Ricci and Levi-Civita,

$$(14.11)\quad \begin{aligned} T^{lm\ldots n}_{ij\ldots k,p,q} - T^{lm\ldots n}_{ij\ldots k,q,p} = {} & T^{\alpha m\ldots n}_{ij\ldots k}\, B^l_{\alpha pq} + \cdots + T^{lm\ldots\alpha}_{ij\ldots k}\, B^n_{\alpha pq} \\ & - T^{lm\ldots n}_{\alpha j\ldots k}\, B^\alpha_{ipq} - \cdots - T^{lm\ldots n}_{ij\ldots\alpha}\, B^\alpha_{kpq}. \end{aligned}$$

Using the tensors D and E (14.11) becomes

$$(14.12)\quad T^{lm\ldots n}_{ij\ldots k,p,q} - T^{lm\ldots n}_{ij\ldots k,q,p} = T^{\alpha\beta\ldots\gamma}_{ij\ldots k}\, B^\tau_{\sigma pq}\, D^{\sigma lm\ldots n}_{\alpha\tau\beta\ldots\gamma} - T^{lm\ldots n}_{\alpha\beta\ldots\gamma}\, B^\sigma_{\tau pq}\, E^{\alpha\tau\beta\ldots\gamma}_{\sigma ij\ldots k}.$$

This identity may be generalized by combining identities of the type (13.12). For example,

$$(14.13)\quad T_{i,p,qr} - T_{i,q,pr} = T_{\alpha,p}\, A^\alpha_{iqr} - T_{\alpha,q}\, A^\alpha_{ipr} - T_{\alpha,r}\, B^\alpha_{ipq} - T_\alpha\, B^\alpha_{ipq,r}.$$

15. **Homogeneous first integrals.** A homogeneous first integral of the kth degree of the differential equations (2.1) is an equation of the form

$$(15.1)\qquad a_{\alpha\beta\ldots\gamma}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\cdots\frac{dx^\gamma}{ds} = \text{constant}$$

which holds along every path. From the equations for the transformation of dx^i/ds it follows that the functions $a_{\alpha\beta\ldots\gamma}$ are the components of a covariant tensor. We shall now derive some general theorems about the conditions

under which a tensor $a_{\alpha\beta\ldots\gamma}$ gives rise to a first integral. The first of these theorems is given by Ricci and Levi-Civita for the case of the Riemann geometry in Chapter 5 of their *Méthodes de calcul différentiel absolu.*

If we differentiate (15.1) with respect to s we have

$$\frac{d a_{\alpha\beta\ldots\gamma}}{ds}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\cdots\frac{dx^\gamma}{ds}+a_{\alpha\beta\ldots\gamma}\frac{d^2x^\alpha}{ds^2}\frac{dx^\beta}{ds}\cdots\frac{dx^\gamma}{ds}+\cdots=0.$$

At the origin of the normal coördinates this equation becomes

$$\frac{\partial a_{\alpha\beta\ldots\gamma}}{\partial y^\sigma}\xi^\alpha\xi^\beta\cdots\xi^\gamma\xi^\sigma=0,$$

owing to the equation (7.1). We may also write

$$a_{\alpha\beta\ldots\gamma,\sigma}\xi^\alpha\xi^\beta\cdots\xi^\gamma\xi^\sigma=0 \tag{15.2}$$

where $a_{\alpha\beta\ldots\gamma,\sigma}$ is the covariant derivative of $a_{\alpha\beta\ldots\gamma}$. The substitution involved in obtaining the last equation is permissible on account of (10.1) which holds at the origin of the normal coördinates. The identity

$$P(a_{\alpha\beta\ldots\gamma,\sigma})=0 \tag{15.3}$$

where P indicates the sum of the terms obtained from the one inside the parenthesis by cyclic permutation of the subscripts, is therefore a necessary condition for the existence of the integral (15.1). Owing to its tensor character (15.3) has validity in all coördinate systems.

To show that (15.3) is also sufficient for the existence of the first integral, let this equation be satisfied by the symmetric tensor $a_{ij\ldots k}$, and express it in its expanded form

$$P\left(\frac{\partial a_{ij\ldots k}}{\partial x^l}-\Gamma^\alpha_{il}a_{\alpha j\ldots k}-\Gamma^\alpha_{jl}a_{i\alpha\ldots k}-\cdots-\Gamma^\alpha_{kl}a_{ij\ldots\alpha}\right)=0.$$

If we consider this equation referred to a normal coördinate system and multiply by $(dy^i/ds)(dy^j/ds)\cdots(dy^k/ds)(dy^l/ds)$, we obtain

$$\frac{d a_{\alpha\beta\ldots\gamma}}{ds}\frac{dy^\alpha}{ds}\frac{dy^\beta}{ds}\cdots\frac{dy^\gamma}{ds}=0, \tag{15.4}$$

on making use of the equation (7.6). Since the derivatives dy^i/ds in (15.4) are constant along any particular path, it follows that

$$a_{\alpha\beta\ldots\gamma}\frac{dy^\alpha}{ds}\frac{dy^\beta}{ds}\cdots\frac{dy^\gamma}{ds} = \text{constant}$$

along any particular path. In consequence of the tensor character of $a_{ij\ldots k}$, we have in general coördinates

$$a_{\alpha\beta\ldots\gamma}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\cdots\frac{dx^\gamma}{ds} = \text{constant.}$$

Hence, *A necessary and sufficient condition for the existence of a homogeneous first integral of the kth degree is that a symmetric covariant tensor of the kth order $a_{ij\ldots k}$ exist which satisfies* (15.3).

If a symmetric tensor $b_{ij\ldots k}$ and function $\varphi(x^1, x^2, \ldots, x^n)$ exist which satisfy the equations

$$P(b_{ij\ldots k,l}) = P(b_{ij\ldots k}\varphi_l), \qquad \varphi_l = \frac{\partial\varphi}{\partial x^l}, \tag{15.5}$$

where $b_{ij\ldots k,l}$ is the covariant derivative of $b_{ij\ldots k}$, a function ψ can be chosen so that the equation

$$P[(\psi b_{ij\ldots k})_l] = 0 \tag{15.6}$$

is satisfied. The bracket contains the covariant derivative of $\psi b_{ij\ldots k}$ with respect to x^l. In fact, we have

$$\begin{aligned} P[(\psi b_{ij\ldots k})_l] & \\ &= P(\psi_l b_{ij\ldots k} + \psi b_{ij\ldots k,l}) \\ &= \psi P\left[b_{ij\ldots k}\left(\frac{\partial \log\psi}{\partial x^l} + \frac{\partial\varphi}{\partial x^l}\right)\right]. \end{aligned}$$

Hence (15.6) is satisfied if we put

$$\psi = e^{-\varphi}.$$

Therefore if (15.5) is satisfied a first integral exists which is given by $e^{-\varphi}\, b_{ij\ldots k}$. That (15.5) is a necessary condition is immediate, for, as we have seen, if $b_{ij\ldots k}$ furnishes a first integral (15.5) is satisfied with $\varphi =$ constant.

Hence, *A necessary and sufficient condition for the existence of a covariant tensor $a_{ij\ldots k}$ which satisfies* (15.3) *is that a covariant tensor $b_{ij\ldots k}$ and function φ exist which satisfy* (15.5). *If the tensor $b_{ij\ldots k}$ and function φ exist, then*

$$a_{ij\ldots k} = e^{-\varphi}\, b_{ij\ldots k}.$$

A particular case of (15.3) is

$$a_{ij\ldots k,l} = 0, \tag{15.7}$$

where $a_{ij\ldots k,l}$ is the covariant derivative of $a_{ij\ldots k}$. In a manner similar to the above it can then be shown that

$$b_{ij\ldots k,l} = b_{ij\ldots k}\, \varphi_l, \qquad \varphi_l = \frac{\partial \varphi}{\partial x^l} \tag{15.8}$$

is a necessary and sufficient condition for the existence of a first integral which satisfies (15.7), and that this integral is given by $a_{ij\ldots k} = e^{-\varphi}\, b_{ij\ldots k}$.

Hence, *A necessary and sufficient condition for the existence of a covariant tensor $a_{ij\ldots k}$ which satisfies* (15.7) *is that a covariant tensor $b_{ij\ldots k}$ and function φ exist which satisfy* (15.8). *If the tensor $b_{ij\ldots k}$ and function φ exist then*

$$a_{ij\ldots k} = e^{-\varphi}\, b_{ij\ldots k}.$$

The equation (14.12) provides a new statement of this last theorem. If the tensor $b_{ij\ldots k}$ satisfies (15.8) we obtain by covariant differentiation

$$\begin{aligned} b_{ij\ldots k,l,m} &= b_{ij\ldots k,m}\, \varphi_l + b_{ij\ldots k}\, \varphi_{l,m} \\ &= b_{ij\ldots k}\, (\varphi_l\, \varphi_m + \varphi_{l,m}). \end{aligned}$$

Hence,

$$b_{ij\ldots k,l,m} - b_{ij\ldots k,m,l} = 0.$$

From (14.12) we then have

$$b_{\alpha\beta\ldots\gamma}\, B^{\sigma}_{\tau lm}\, E^{\alpha\tau\beta\ldots\gamma}_{\sigma ij\ldots k} = 0. \tag{15.9}$$

Conversely, if the tensor $b_{ij\ldots k}$ and vector φ_l satisfy (15.9) and

$$b_{ij\ldots k,l} = b_{ij\ldots k}\,\varphi_l, \tag{15.10}$$

where $b_{ij\ldots k,l}$ is the covariant derivative of $b_{ij\ldots k}$, we have

$$b_{ij\ldots k,l,m} = b_{ij\ldots k}\;(\varphi_l\,\varphi_m + \varphi_{l,m}),$$

and

$$b_{ij\ldots k,l,m} - b_{ij\ldots k,m,l} = b_{ij\ldots k}\;(\varphi_{l,m} - \varphi_{m,l}).$$

Since $b_{ij\ldots k}$ satisfies (15.9)

$$b_{ij\ldots k,l,m} - b_{ij\ldots k,m,l} = 0,$$

so that

$$\varphi_{l,m} - \varphi_{m,l} = 0$$

or

$$\frac{\partial \varphi_l}{\partial x^m} = \frac{\partial \varphi_m}{\partial x^l}$$

and this last equation is the condition that φ_l be the gradient of a scalar function $\varphi(x^1, x^2, \ldots, x^n)$, i. e.,

$$\varphi_l = \frac{\partial \varphi}{\partial x^l}.$$

Hence, *A necessary and sufficient condition for the existence of a covariant tensor* $a_{ij\ldots k}$ *which satisfies* (15.7) *is that a covariant tensor* $b_{ij\ldots k}$ *and vector* φ_l *exist which satisfy* (15.9) *and* (15.10). *If the tensor* $b_{ij\ldots k}$ *and vector* φ_l *exist, then*

$$a_{ij\ldots k} = e^{-\varphi}\,b_{ij\ldots k}.$$

16. Algebraic condition for existence of first integrals of a particular class. We shall now derive a condition on the functions Γ for the existence of a homogeneous first integral of the kth degree which satisfies the particular condition (15.7). The condition is to involve only the algebraic consistency of a set of tensor equations formed from the functions Γ. If the

covariant tensor of the kth order $a_{ij\dots k}$ satisfies the condition (15.7) it follows by (14.12) that $a_{ij\dots k}$ will satisfy a sequence of equations of the form

$$
\begin{aligned}
&a_{\alpha\beta\dots\gamma}\, D^{\alpha\beta\dots\gamma}_{ij\dots klm} = 0,\\
&a_{\alpha\beta\dots\gamma}\, D^{\alpha\beta\dots\gamma}_{ij\dots klm, r_1} = 0,\\
&a_{\alpha\beta\dots\gamma}\, D^{\alpha\beta\dots\gamma}_{ij\dots klm, r_1, r_2} = 0,\\
&\dots\dots\dots\dots\\
(16.1)\qquad &\dots\dots\dots\dots\\
&\dots\dots\dots\dots\\
&a_{\alpha\beta\dots\gamma}\, D^{\alpha\beta\dots\gamma}_{ij\dots klm, r_1, r_2, \dots, r_n} = 0,\\
&\dots\dots\dots\dots\\
&\dots\dots\dots\dots\\
&\dots\dots\dots\dots
\end{aligned}
$$

where

$$D^{\alpha\beta\dots\gamma}_{ij\dots klm} = B^{\sigma}_{\tau lm}\, E^{\alpha\tau\beta\dots\gamma}_{\sigma ij\dots k}$$

and $D^{\alpha\beta\dots\gamma}_{ij\dots klm, r_1, r_2, \dots, r_n}$ represents the nth covariant derivative of $D^{\alpha\beta\dots\gamma}_{ij\dots klm}$. The algebraic consistency of the equations (16.1) is a necessary condition on the Γ's for the existence of the homogeneous first integral of the kth degree which satisfies (15.7).

The algebraic solutions of the equations (16.1) possess a tensor character. For let $a_{ij\dots k}$ represent an algebraic solution of (16.1). Under a general transformation of coördinates the first set of equations of (16.1) becomes

$$a_{\alpha\beta\dots\gamma}\, \overline{D}^{\alpha\beta\dots\gamma}_{ij\dots klm} = 0, \tag{16.2}$$

where $\overline{D}^{\alpha\beta\dots\gamma}_{ij\dots klm}$ is defined by the equations of transformation

$$D^{ij\dots k}_{pq\dots rst} = \frac{\partial x^i}{\partial \overline{x}^\alpha} \cdots \frac{\partial x^k}{\partial \overline{x}^\beta} \frac{\partial \overline{x}^\gamma}{\partial x^p} \cdots \frac{\partial \overline{x}^\delta}{\partial x^t}\, \overline{D}^{\alpha\dots\beta}_{\gamma\dots\delta} \tag{16.3}$$

and $\bar{a}_{\alpha\beta\ldots\gamma}$ represents an algebraic solution of (16.2). Substituting (16.3) in the first set of equations of (16.1) we obtain

$$
(16.4) \qquad a_{\varepsilon\ldots\omega} \frac{\partial x^{\varepsilon}}{\partial \bar{x}^{\mu}} \cdots \frac{\partial x^{\omega}}{\partial \bar{x}^{\nu}} \frac{\partial \bar{x}^{\gamma}}{\partial x^{p}} \cdots \frac{\partial \bar{x}^{\delta}}{\partial x^{t}} \bar{D}^{\mu\ldots\nu}_{\gamma\ldots\delta} = 0.
$$

If we multiply (16.4) by $(\partial x^{p}/\partial \bar{x}^{i})(\partial x^{q}/\partial \bar{x}^{j}) \cdots (\partial x^{t}/\partial \bar{x}^{m})$ and sum for $(p, q, \ldots, t)$, then

$$
a_{\varepsilon\ldots\omega} \frac{\partial x^{\varepsilon}}{\partial \bar{x}^{\mu}} \cdots \frac{\partial x^{\omega}}{\partial \bar{x}^{\nu}} \bar{D}^{\mu\ldots\nu}_{ij\ldots klm} = 0
$$

and a comparison of these equations with (16.2) shows that a solution of (16.2) is given by

$$
\bar{a}_{ij\ldots k} = a_{\varepsilon\ldots\omega} \frac{\partial x^{\varepsilon}}{\partial \bar{x}^{i}} \cdots \frac{\partial x^{\omega}}{\partial \bar{x}^{k}}.
$$

While we have considered the first set of equations of (16.1) a similar result would have been obtained with regard to any other set. Hence the algebraic solutions of (16.1) are tensors and it is consequently permissible to form the covariant derivative of these solutions as we shall do in the later work.

Let us now assume the algebraic consistency of the equations (16.1) and suppose that the first system of these equations admits a set of fundamental solutions denoted by $b^{(p)}_{ij\ldots k}$, $p = 1, 2, \ldots, s$. The general solution of this system of equations can then be expressed as a linear combination of the fundamental solutions $b^{(p)}_{ij\ldots k}$ with arbitrary functional coefficients. We next consider the first and second systems of equations (16.1) and suppose that these equations have a fundamental set of solutions $c^{(p)}_{ij\ldots k}$, $p = 1, 2, \ldots, t$, in which of course $s \geqq t$. If $s = t$ then $c^{(p)}_{ij\ldots k}$, $p = 1, 2, \ldots, t$ will furnish a fundamental set of solutions of the first system of equations which satisfies the second system. If $s > t$ we consider the first three systems of equations, which we may suppose to have a fundamental set of solutions $d^{(p)}_{ij\ldots k}$, $p = 1, 2, \ldots, u$, with the condition $t \geqq u$. In case $t = u$ then $d^{(p)}_{ij\ldots k}$, $p = 1, 2, \ldots, u$, will furnish a fundamental set of solutions of the first two systems of equations which satisfies the third system. By proceeding in this way we shall finally come to a point where the first N systems of equations of (16.1) will admit a fundamental set of solutions which satisfies the system immediately following in the sequence. Hence to say that the equations (16.1) are

algebraically consistent implies that there is a number N such that the first N systems of equations (16.1) admit a fundamental set of solutions $a^{(p)}_{ij\ldots k}$, $p = 1, 2, \ldots, s$, which satisfies the equation

$$a_{\alpha\beta\ldots\gamma} D^{\alpha\beta\ldots\gamma}_{ij\ldots klm, r_1, r_2, \ldots, r_N} = 0.$$

The general solution of the first N systems of equations is then

$$a_{ij\ldots k} = \varphi^{(\alpha)} a^{(\alpha)}_{ij\ldots k} \qquad (\alpha = 1, 2, \ldots, s), \tag{16.5}$$

where the expression on the right is summed for α, and $\varphi^{(\alpha)}$ is an arbitrary function of $(x^1, x^2, \ldots, x^n)$.

Before proceeding further with the general case let us consider the particular case where the first system of equations (16.1) has a unique solution $a_{ij\ldots k}$ which satisfies the second system, i. e.,

$$a_{\alpha\beta\ldots\gamma} D^{\alpha\beta\ldots\gamma}_{ij\ldots klm, r_1} = 0. \tag{16.6}$$

Under these conditions a homogeneous first integral of the kth degree will exist whose covariant derivative vanishes. For if we differentiate the first system of equations (16.1) covariantly we obtain, on account of (16.6),

$$a_{\alpha\beta\ldots\gamma, r} D^{\alpha\beta\ldots\gamma}_{ij\ldots klm} = 0. \tag{16.7}$$

where $a_{\alpha\beta\ldots\gamma,r}$ is the covariant derivative of $a_{\alpha\beta\ldots\gamma}$. Since (16.7) possesses a unique solution $a_{ij\ldots k}$, it follows that

$$a_{ij\ldots k, l} = \varphi_l \, a_{ij\ldots k}$$

in which φ_l is a covariant vector. The above statement then follows from the last theorem of § 15.

Going back to the general case let us substitute one of the fundamental solutions $a^{(p)}_{ij\ldots k}$ in the equations of the sequence (16.1) through the $(N+1)$th. We may then differentiate these equations covariantly so as to obtain the following:

$$
\begin{aligned}
&a^{(p)}_{\alpha\beta\ldots\gamma,r}\, D^{\alpha\beta\ldots\gamma}_{ij\ldots klm} = 0,\\
&a^{(p)}_{\alpha\beta\ldots\gamma,r}\, D^{\alpha\beta\ldots\gamma}_{ij\ldots klm,r_1} = 0,\\
(16.8)\qquad &\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\\
&a^{(p)}_{\alpha\beta\ldots\gamma,r}\, D^{\alpha\beta\ldots\gamma}_{ij\ldots klm,r_1,r_2,\ldots,r_{N-1}} = 0,
\end{aligned}
$$

where $a^{(p)}_{\alpha\beta\ldots\gamma,r}$ is the covariant derivative of $a^{(p)}_{\alpha\beta\ldots\gamma}$. Since $a^{(p)}_{ij\ldots k,l}$ is a solution of (16.8) it may be expressed linearly in terms of the fundamental solutions of these equations. Hence

$$
a^{(p)}_{ij\ldots k,l} = \lambda^{(p\alpha)}_l\, a^{(\alpha)}_{ij\ldots k}, \tag{16.9}
$$

where the expression on the right is summed for α, and the λ's are covariant vectors. Since $a^{(p)}_{ij\ldots k}$ satisfies the first system of the sequence (16.1),

$$
a^{(p)}_{ij\ldots k,l,m} - a^{(p)}_{ij\ldots k,m,l} = 0. \tag{16.10}
$$

If $a^{(p)}_{ij\ldots k,l}$ as given by (16.9) be differentiated covariantly and substituted in (16.10) there is obtained the following condition on the λ's:

$$
\frac{\partial \lambda^{(pq)}_k}{\partial x^l} - \frac{\partial \lambda^{(pq)}_l}{\partial x^k} + \lambda^{(p\alpha)}_k \lambda^{(\alpha q)}_l - \lambda^{(p\alpha)}_l \lambda^{(\alpha q)}_k = 0. \tag{16.11}
$$

If we substitute (16.5) in (15.7) we see that it will be satisfied if a set of functions $\varphi^{(p)}$, $p = 1, 2, \ldots, s$ can be chosen so as to satisfy the equations

$$
\frac{\partial \varphi^{(p)}}{\partial x^k} + \varphi^{(\alpha)} \lambda^{(\alpha p)}_k = 0. \tag{16.12}
$$

Such a set of functions can be chosen, for in consequence of (16.11) these equations are completely integrable. This set of functions $\varphi^{(\alpha)}$ will determine

according to (16.5) a covariant tensor of the kth degree $a_{ij\ldots k}$ whose covariant derivative vanishes.

Hence, *a necessary and sufficient condition for the existence of a homogeneous first integral of the kth degree* $a_{ij\ldots k}$ *which satisfies* (15.7) *is that there exists a number* N *such that the first* N *systems of equations* (16.1) *admit a fundamental set of* s *solutions* ($s \geqq 1$) *which satisfy the* $(N+1)$*th system of equations.*

17. Special cases. The theorems of the last two sections have some interesting applications in the linear and quadratic cases. It is natural to define a field of parallel covariant vectors by means of a set of functions h_i such that

$$h_{i,j} = 0. \tag{17.1}$$

For this means that if normal coördinates are introduced with origin at an arbitrary point, we have at this point

$$\frac{dh_i}{ds} = \frac{\partial h_i}{\partial y^\alpha}\frac{dy^\alpha}{ds} = 0. \tag{17.2}$$

By the third theorem in italics in § 15, a necessary and sufficient condition for the existence of a field of parallel covariant vectors is the existence of a function φ and vector A_i such that*

$$A_{i,j} = A_i\,\varphi_j, \quad \varphi_j = \frac{\partial\varphi}{\partial x^j}. \tag{17.3}$$

The last theorem of § 15 now shows that a necessary and sufficient condition for a field of parallel covariant vectors h_i is that a covariant vector A_i exist which satisfies the equations

$$A_{i,j} = A_i\,\varphi_j, \tag{17.4}$$

$$A_\alpha B^\alpha_{ijk} = 0, \tag{17.5}$$

where φ_j is a covariant vector and $A_{i,j}$ is the covariant derivative of A_i. The theorem of § 16 shows that a necessary and sufficient condition for a field of

* Eisenhart, Proceedings of the National Academy of Sciences, vol. 8 (1922), pp. 207–212, defines A_i as a field of parallel vectors, and finds the condition (17.5) for their existence.

parallel covariant vectors is that there exists a number N such that the first N sets of equations of the sequence

$$
\begin{aligned}
A_\alpha B^\alpha_{ijk} &= 0, \\
A_\alpha B^\alpha_{ijk,l} &= 0, \\
A_\alpha B^\alpha_{ijk,l,m} &= 0, \qquad (17.6)\\
&\cdots\cdots \\
&\cdots\cdots \\
&\cdots\cdots
\end{aligned}
$$

admit a fundamental set of s solutions ($s \geqq 1$) which satisfy the $(N+1)$th set. In particular a sufficient condition is obtained if the first system of equations of (17.6) be algebraically consistent and all their solutions satisfy the second system of these equations.

Going now to the quadratic case we see from the third theorem in italics in § 15 that the condition on the functions Γ for the geometry of paths to become a Riemann geometry is that a tensor g_{ij} exist such that

$$g_{ij,k} = g_{ij}\,\varphi_k, \quad \varphi_k = \frac{\partial \varphi}{\partial x^k}. \tag{17.7}$$

The equation (17.7) without the condition that the vector φ_k be the gradient of a scalar function gives the geometry upon which Weyl bases his electromagnetic and gravitational theory, for this equation is equivalent to the equation (2.10). By the last theorem of § 15, the condition (17.7) can be written

$$g_{ij,k} = g_{ij}\,\varphi_k, \tag{17.8}$$

$$g_{\alpha j} B^\alpha_{ikl} + g_{i\alpha} B^\alpha_{jkl} = 0. \tag{17.9}$$

This shows furthermore that a necessary and sufficient condition for the Weyl geometry to become the Riemann geometry is that the tensor g_{ij} satisfy (17.9).

The theorem of § 16 shows that a necessary and sufficient condition for the geometry of paths to become a Riemann geometry is that there exists a number N such that the first N systems of equations of the following sequence

admit a fundamental set of s solutions ($s \geqq 1$) which satisfy the $(N+1)$st system of equations:

$$\begin{aligned} g_{\alpha j} B^{\alpha}_{ikl} + g_{i\alpha} B^{\alpha}_{jkl} &= 0, \\ g_{\alpha j} B^{\alpha}_{ikl,m} + g_{i\alpha} B^{\alpha}_{jkl,m} &= 0, \\ g_{\alpha j} B^{\alpha}_{ikl,m,n} + g_{i\alpha} B^{\alpha}_{jkl,m,n} &= 0, \\ \cdots\cdots & \\ \cdots\cdots & \\ \cdots\cdots & \end{aligned} \tag{17.10}$$

In particular* we have that a sufficient condition for the geometry of paths to become a Riemann geometry is that the equations

$$g_{\alpha j} B^{\alpha}_{ikl} + g_{i\alpha} B^{\alpha}_{jkl} = 0$$

be algebraically consistent and that all their solutions satisfy

$$g_{\alpha j} B^{\alpha}_{ikl,m} + g_{i\alpha} B^{\alpha}_{jkl,m} = 0.$$

18. **The homogeneous linear first integral.** From the first theorem of § 15 it follows that a necessary and sufficient condition for the covariant vector h_i to furnish a linear first integral,

$$h_{\alpha} \frac{dx^{\alpha}}{ds} = \text{constant}, \tag{18.1}$$

is that the equation

$$h_{i,j} + h_{j,i} = 0 \tag{18.2}$$

be satisfied, i. e., the covariant derivative $h_{i,j}$ must be skew symmetric in the indices i and j. The equations (14.12) show that

$$h_{i,j,k} - h_{i,k,j} = h_{\alpha} B^{\alpha}_{ikj}. \tag{18.3}$$

* Eisenhart and Veblen, Proceedings of the National Academy of Sciences, vol. 8 (1922), pp. 19–23.

By (18.2) these give rise to

$$
\begin{aligned}
h_{i,j,k} + h_{k,i,j} &= h_\alpha B^\alpha_{ikj}, \\
h_{k,i,j} + h_{j,k,i} &= h_\alpha B^\alpha_{kji}, \qquad (18.4)\\
h_{j,k,i} + h_{i,j,k} &= h_\alpha B^\alpha_{jik}.
\end{aligned}
$$

If we add these three equations we obtain

$$
h_{i,j,k} + h_{j,k,i} + h_{k,i,j} = 0. \qquad (18.5)
$$

Combining (18.5) with the second equation of (18.4), we have also

$$
h_{i,j,k} = h_\alpha B^\alpha_{kij}. \qquad (18.6)
$$

These are integrability conditions obtained by consideration of second derivatives. In order to obtain those involving third derivatives we use (13.12), which with (18.2) gives

$$
\begin{aligned}
h_{i,pqr} + h_{p,iqr} &= 2h_{\alpha,q} A^\alpha_{ipr} + 2h_{\alpha,r} A^\alpha_{ipq} + 2h_\alpha A^\alpha_{ipqr}, \\
h_{q,ipr} + h_{i,qpr} &= 2h_{\alpha,p} A^\alpha_{qir} + 2h_{\alpha,r} A^\alpha_{qip} + 2h_\alpha A^\alpha_{qipr}, \qquad (18.7)\\
h_{p,qir} + h_{q,pir} &= 2h_{\alpha,i} A^\alpha_{pqr} + 2h_{\alpha,r} A^\alpha_{pqi} + 2h_\alpha A^\alpha_{pqir}.
\end{aligned}
$$

If we add the first two of these equations and subtract the third, we obtain

$$
\begin{aligned}
h_{i,pqr} = {} & h_{\alpha,p} A^\alpha_{iqr} + h_{\alpha,q} A^\alpha_{ipr} - h_{\alpha,i} A^\alpha_{pqr} \\
& + h_{\alpha,r}(A^\alpha_{ipq} + A^\alpha_{qip} - A^\alpha_{pqi}) + h_\alpha(A^\alpha_{ipqr} + A^\alpha_{qipr} - A^\alpha_{pqir}). \qquad (18.8)
\end{aligned}
$$

Now interchange the indices q and r and subtract the resulting equation from this one. We obtain

$$(18.9) \quad h_{\alpha,i} B^{\alpha}_{prq} + h_{\alpha,p} B^{\alpha}_{iqr} + h_{\alpha,q} B^{\alpha}_{rpi} + h_{\alpha,r} B^{\alpha}_{qip} + h_{\alpha} (B^{\alpha}_{qip,r} + B^{\alpha}_{rpi,q}) = 0.$$

If we collect the terms in the equation (18.9) we have

$$(18.10) \qquad h_{\alpha} C^{\alpha} + h_{\alpha,\beta} D^{\alpha\beta} = 0,$$

where

$$C^{\alpha} = B^{\alpha}_{\gamma\delta\varepsilon,\mu} (\delta^{\gamma}_{k} \delta^{\delta}_{i} \delta^{\varepsilon}_{j} \delta^{\mu}_{l} + \delta^{\gamma}_{l} \delta^{\delta}_{j} \delta^{\varepsilon}_{i} \delta^{\mu}_{k}),$$

$$D^{\alpha\beta} = B^{\alpha}_{\gamma\delta\varepsilon} (\delta^{\beta}_{i} \delta^{\gamma}_{j} \delta^{\delta}_{l} \delta^{\varepsilon}_{k} + \delta^{\beta}_{j} \delta^{\gamma}_{i} \delta^{\delta}_{k} \delta^{\varepsilon}_{l} + \delta^{\beta}_{k} \delta^{\gamma}_{l} \delta^{\delta}_{j} \delta^{\varepsilon}_{i} + \delta^{\beta}_{l} \delta^{\gamma}_{k} \delta^{\delta}_{i} \delta^{\varepsilon}_{j}).$$

C^{α} is a tensor which is contravariant of the first order and covariant of the fourth, $D^{\alpha\beta}$ is a tensor which is contravariant of the second order and covariant of the fourth. The covariant indices of these tensors have been omitted for simplicity. If we differentiate (18.10) covariantly, we obtain

$$h_{\alpha,i} C^{\alpha} + h_{\alpha} C^{\alpha}_{,i} + h_{\alpha,\beta,i} D^{\alpha\beta} + h_{\alpha,\beta} D^{\alpha\beta}_{,i} = 0,$$

and this becomes

$$(18.11) \qquad h_{\alpha,i} C^{\alpha} + h_{\alpha} C^{\alpha}_{,i} + h_{\gamma} B^{\gamma}_{i\alpha\beta} D^{\alpha\beta} + h_{\alpha,\beta} D^{\alpha\beta}_{,i} = 0$$

when we make the substitution (18.6). This equation may be written in an abbreviated form as follows:

$$(18.12) \qquad h_{\alpha} C^{\alpha}_{1} + h_{\alpha,\beta} D^{\alpha\beta}_{1} = 0.$$

Covariant differentiation of (18.12) will give rise to a new equation which can in its turn be abbreviated to the form (18.10), this process requiring the use of (18.6) to eliminate $h_{\alpha,\beta,\gamma}$. Continuing in this way we obtain an infinite sequence of equations. For the purpose of convenient reference we shall write this sequence with the equation (18.2) as the first member:

$$
(18.13)\qquad
\begin{aligned}
h_{\alpha,\beta}+h_{\beta,\alpha} &= 0,\\
h_\alpha C^\alpha + h_{\alpha,\beta} D^{\alpha\beta} &= 0,\\
h_\alpha C^\alpha_1 + h_{\alpha,\beta} D^{\alpha\beta}_1 &= 0,\\
&\cdots\cdots\cdots\\
&\cdots\cdots\cdots\\
&\cdots\cdots\cdots\\
h_\alpha C^\alpha_n + h_{\alpha,\beta} D^{\alpha\beta}_n &= 0,\\
&\cdots\cdots\cdots\\
&\cdots\cdots\cdots\\
&\cdots\cdots\cdots.
\end{aligned}
$$

The algebraic consistency of this set of equations, regarded as equations for the determination of h_i and $h_{i,j}$, is a necessary condition for the existence of a first integral h_i. Hence as in § 16 there must be a value of N such that the first $N+1$ sets of equations admit a fundamental set of solutions $h_i^{(p)}$, $h_{ij}^{(p)}$ $(p=1,2,\ldots,s)$ each of which will satisfy the system of equations next following in the sequence (18.13). This necessary condition turns out also to be a sufficient condition.

Before proving this in general, let us consider the special case in which $N = 1$ and $s = 1$. In this case (18.2) and (18.10) are consistent and possess a unique algebraic solution consisting of a set of functions h_i, $i = 1, 2, \ldots, n$, and a set of functions h_{ij}, $i, j = 1, 2, \ldots, n$, which satisfy (18.11). It may now be shown that the solutions h_i and h_{ij} are tensors, so that it is possible to substitute these quantities in the equation (18.10) and differentiate it covariantly. Doing this we obtain

$$
(18.14)\qquad h_{\alpha,i}\, C^\alpha + h_\alpha\, C^\alpha_{,i} + h_{\alpha\beta,i}\, D^{\alpha\beta} + h_{\alpha\beta}\, D^{\alpha\beta}_{,i} = 0.
$$

If we subtract (18.14) from (18.11) with h_{ij} replacing $h_{i,j}$, we obtain

$$
(18.15)\qquad (h_{\alpha i} - h_{\alpha,i})\, C^\alpha + (h_\gamma B^\gamma_{i\alpha\beta} - h_{\alpha\beta,i})\, D^{\alpha\beta} = 0.
$$

In (18.15) the coefficient of $D^{\alpha\beta}$ is skew symmetric in the indices α and β. By hypothesis, the solution of (18.15) and (18.2) is unique and consequently the solution $(h_{\alpha i} - h_{\alpha, i})$, $(h_\gamma B^\gamma_{i\alpha\beta} - h_{\alpha\beta, i})$ can only differ from the solution h_α, $h_{\alpha\beta}$ by a factor of multiplication. Hence

$$h_{ij} - h_{i,j} = \varphi_j h_i, \tag{18.16}$$

$$h_\alpha B^\alpha_{kij} - h_{i,j,k} = \varphi_k h_{ij}, \tag{18.17}$$

where φ_i is a covariant vector. If we differentiate (18.16) covariantly, obtaining

$$h_{ij,k} - h_{i,j,k} = \varphi_{j,k} h_i + \varphi_j h_{i,k},$$

and from this form the expression

$$(\varphi_{j,k} - \varphi_{k,j}) h_i = h_{ij,k} - h_{ik,j} + h_{i,k,j} - h_{i,j,k} + \varphi_k h_{i,j} - \varphi_j h_{i,k},$$

we find on substituting the equations (18.16) and (18.17) in the right member of this equation that it vanishes identically. The functions h_i are not all identically zero, for if so it would follow by (18.16) that the functions h_{ij} are also identically zero, contrary to the assumption that (18.2) and (18.10) are algebraically consistent. Hence

$$\varphi_{j,k} - \varphi_{k,j} = 0.$$

The vector φ_i is therefore the gradient of a scalar function φ, i.e.,

$$\varphi_i = \frac{\partial \varphi}{\partial x^i}.$$

Now we shall have a first integral if a function ψ exists such that

$$(\psi h_i)_j = \psi h_{ij}, \tag{18.18}$$

where $(\psi h_i)_j$ denotes the covariant derivative of ψh_i. For if this equation is satisfied, ψh_i will be a covariant vector satisfying (18.2) and hence will give a first integral. Expanding (18.18)

$$\frac{\partial \psi}{\partial x^j} h_i + \psi h_{i,j} = \psi h_{ij},$$

or

$$h_{ij} - h_{i,j} = \varPsi_j h_i, \tag{18.19}$$

where

$$\varPsi_j = \frac{\partial \varPsi}{\partial x^j} \text{ and } \varPsi = \log \psi.$$

The gradient $\varPsi_j$ is a covariant vector and consequently (18.19) will be satisfied if we put

$$\varPsi = \varphi.$$

Hence, a sufficient condition for the existence of a linear first integral is that (18.2) and (18.10) be algebraically consistent and that they possess a unique solution which satisfies (18.11).

Let us now return to the general case and assume that there is a value of N such that the first $N+1$ systems of equations (18.13) admit a fundamental system of solutions $h_i^{(p)}$, $h_{ij}^{(p)}$, $p = 1, 2, \ldots, s$, each of which satisfies the system of equations immediately following in the sequence. By the same argument as before, $h_i^{(p)}$ and $h_{ij}^{(p)}$ are tensors for all values of p. The general solution of the first $N+1$ systems of equations is then

$$h_i = \varphi^{(\alpha)} h_i^{(\alpha)}, \tag{18.20}$$

$$h_{ij} = \varphi^{(\alpha)} h_{ij}^{(\alpha)}, \tag{18.21}$$

where the terms on the right are summed for α from $\alpha = 1$ to $\alpha = s$. If we differentiate the equation

$$h_\alpha^{(p)} C^\alpha + h_{\alpha\beta}^{(p)} D^{\alpha\beta} = 0 \tag{18.22}$$

covariantly, and subtract it from the equation

$$h_{\alpha}^{(p)} C_1^{\alpha} + h_{\alpha\beta}^{(p)} D_1^{\alpha\beta} = 0, \tag{18.23}$$

we obtain

$$(h_{\alpha i}^{(p)} - h_{\alpha,i}^{(p)}) C^{\alpha} + (h_{\gamma}^{(p)} B_{i\alpha\beta}^{\gamma} - h_{\alpha\beta,i}^{(p)}) D^{\alpha\beta} = 0. \tag{18.24}$$

If we next differentiate (18.23) covariantly and subtract it from the equation immediately following in the sequence, we have

$$(h_{\alpha i}^{(p)} - h_{\alpha,i}^{(p)}) C^{\alpha} + (h_{\gamma}^{(p)} B_{i\alpha\beta}^{\gamma} - h_{\alpha\beta,i}^{(p)}) D^{\alpha\beta} = 0.$$

Continuing in this way we obtain the equations

$$\begin{aligned}
&(h_{\alpha i}^{(p)} - h_{\alpha,i}^{(p)}) C^{\alpha} + (h_{\gamma}^{(p)} B_{i\alpha\beta}^{\gamma} - h_{\alpha\beta,i}^{(p)}) D^{\alpha\beta} = 0,\\
&(h_{\alpha i}^{(p)} - h_{\alpha,i}^{(p)}) C_1^{\alpha} + (h_{\gamma}^{(p)} B_{i\alpha\beta}^{\gamma} - h_{\alpha\beta,i}^{(p)}) D_1^{\alpha\beta} = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(h_{\alpha i}^{(p)} - h_{\alpha,i}^{(p)}) C_n^{\alpha} + (h_{\gamma}^{(p)} B_{i\alpha\beta}^{\gamma} - h_{\alpha\beta,i}^{(p)}) D_n^{\alpha\beta} = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.
\end{aligned}$$

The term $(h_{\gamma}^{(p)} B_{i\alpha\beta}^{\gamma} - h_{\alpha\beta,i}^{(p)})$ is skew symmetric in the indices α and β, and we may therefore express the quantities $(h_{\alpha i}^{(p)} - h_{\alpha,i}^{(p)})$, $(h_{\gamma}^{(p)} B_{i\alpha\beta}^{\gamma} - h_{\alpha\beta,i}^{(p)})$ as a linear combination of the particular solutions $h_{\alpha}^{(p)}$, $h_{\alpha\beta}^{(p)}$, $p = 1, 2, \ldots, s$:

$$h_{i,j}^{(p)} - h_{ij}^{(p)} = \lambda_j^{(p\alpha)} h_i^{(\alpha)}, \tag{18.25}$$

$$h_{ij,k}^{(p)} - h_{\alpha}^{(p)} B_{kij}^{\alpha} = \lambda_k^{(p\alpha)} h_{ij}^{(\alpha)}. \tag{18.26}$$

To determine the condition which the covariant vectors $\lambda_k^{(ij)}$ must satisfy we differentiate (18.25) covariantly, obtaining

$$h^{(k)}_{i,p,q} = h^{(k)}_{ip,q} + \lambda^{(k\alpha)}_{p,q}\, h^{(\alpha)}_{i} + \lambda^{(k\alpha)}_{p}\, h^{(\alpha)}_{i,q},$$

or

$$(18.27)\quad h^{(k)}_{i,p,q} = h^{(k)}_{\alpha}\, B^{\alpha}_{qip} + \lambda^{(k\alpha)}_{q}\, h^{(\alpha)}_{ip} + \lambda^{(k\alpha)}_{p,q}\, h^{(\alpha)}_{i} + \lambda^{(k\alpha)}_{p}\, (h^{(\alpha)}_{iq} + \lambda^{(\alpha\beta)}_{q}\, h^{(\beta)}_{i}).$$

If we interchange p and q in (18.27) and subtract these two equations we find that

$$(18.28)\qquad h^{(\alpha)}_{i}\, (\lambda^{(k\alpha)}_{p,q} - \lambda^{(k\alpha)}_{q,p} + \lambda^{(k\beta)}_{p}\, \lambda^{(\beta\alpha)}_{q} - \lambda^{(k\beta)}_{q}\, \lambda^{(\beta\alpha)}_{p}) = 0.$$

We next differentiate (18.26) covariantly,

$$h^{(k)}_{ip,q,r} = h^{(k)}_{\alpha,r}\, B^{\alpha}_{qip} + h^{(k)}_{\alpha}\, B^{\alpha}_{qip,r} + \lambda^{(k\alpha)}_{q,r}\, h^{(\alpha)}_{ip} + \lambda^{(k\alpha)}_{q}\, h^{(\alpha)}_{ip,r},$$

or

$$(18.29)\qquad \begin{aligned} h^{(k)}_{ip,q,r} = {} & (h^{(k)}_{\alpha r} + \lambda^{(k\beta)}_{r}\, h^{(\beta)}_{\alpha})\, B^{\alpha}_{qip} + h^{(k)}_{\alpha}\, B^{\alpha}_{qip,r} + \lambda^{(k\alpha)}_{q,r}\, h^{(\alpha)}_{ip} \\ & + \lambda^{(k\alpha)}_{q}\, (h^{(\alpha)}_{\beta}\, B^{\beta}_{rip} + \lambda^{(\alpha\beta)}_{r}\, h^{(\beta)}_{ip}). \end{aligned}$$

Interchanging r and q in (18.29) and subtracting the two equations,

$$(18.30)\qquad \begin{aligned} & h^{(\alpha)}_{ip}\, (\lambda^{(k\alpha)}_{q,r} - \lambda^{(k\alpha)}_{r,q} + \lambda^{(k\beta)}_{q}\, \lambda^{(\beta\alpha)}_{r} - \lambda^{(k\beta)}_{r}\, \lambda^{(\beta\alpha)}_{q}) \\ & + h^{(k)}_{\alpha}\, (B^{\alpha}_{qip,r} - B^{\alpha}_{rip,q}) + h^{(k)}_{\alpha r}\, B^{\alpha}_{qip} + h^{(k)}_{\alpha q}\, B^{\alpha}_{rpi} + h^{(k)}_{\alpha p}\, B^{\alpha}_{iqr} + h^{(k)}_{\alpha i}\, B^{\alpha}_{prq} = 0. \end{aligned}$$

This equation reduces to

$$(18.31)\qquad h^{(\alpha)}_{ip}\, (\lambda^{(k\alpha)}_{q,r} - \lambda^{(k\alpha)}_{r,q} + \lambda^{(k\beta)}_{q}\, \lambda^{(\beta\alpha)}_{r} - \lambda^{(k\beta)}_{r}\, \lambda^{(\beta\alpha)}_{q}) = 0,$$

since $h^{(\alpha)}_{i}$, $h^{(\alpha)}_{ij}$ is a solution of (18.9). From (18.28) and (18.31) we now deduce that

$$(18.32)\qquad \lambda^{(k\alpha)}_{q,r} - \lambda^{(k\alpha)}_{r,q} + \lambda^{(k\beta)}_{q}\, \lambda^{(\beta\alpha)}_{r} - \lambda^{(k\beta)}_{r}\, \lambda^{(\beta\alpha)}_{q} = 0,$$

for if (18.32) were not satisfied there would be a linear relation among the solutions $h_i^{(k)}$, $h_{ip}^{(k)}$, contrary to the hypothesis that $h_i^{(k)}$, $h_{ip}^{(k)}$, $k = 1, 2, \ldots, s$, is a fundamental set of solutions.

A linear first integral h_i will be determined by (18.20) if $\varphi^{(\alpha)}$ can be chosen so that the equations

$$(18.33) \qquad (\varphi^{(\alpha)} h_i^{(\alpha)})_p = \varphi^{(\alpha)} h_{ip}^{(\alpha)}$$

are satisfied, where the term on the left is the covariant derivative of $\varphi^{(\alpha)} h_i^{(\alpha)}$, for the covariant vector $h_i = \varphi^{(\alpha)} h_i^{(\alpha)}$ will possess a covariant derivative $h_{i,p}$ which is skew symmetric in i and p. Expanding (18.33)

$$(18.34) \qquad \frac{\partial \varphi^{(\alpha)}}{\partial x^p} h_i^{(\alpha)} + \varphi^{(\alpha)} (h_{i,p}^{(\alpha)} - h_{ip}^{(\alpha)}) = 0.$$

From (18.25) we find that the condition on the φ's can be put in the form

$$(18.35) \qquad \frac{\partial \varphi^{(\beta)}}{\partial x^p} + \varphi^{(\alpha)} \lambda_p^{(\alpha\beta)} = 0.$$

The integrability conditions of (18.35) are the equations (18.32) and hence a set of φ's can be found which will satisfy (18.33).

Hence, *a necessary and sufficient condition for the existence of a linear first integral* (18.1) *is that the Γ's be such that there exists a number N such that the first $N+1$ systems of equations* (18.13) *admit a fundamental set of s solutions* ($s \geqq 1$) *which satisfies the* $(N+2)$*nd system of equations.*

19. **The homogeneous quadratic first integral.** A necessary and sufficient condition for the existence of a homogeneous quadratic first integral

$$(19.1) \qquad g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = \text{constant}$$

such that

$$(19.2) \qquad g_{ij} = g_{ji}$$

is that g_{ij} satisfy the condition

$$(19.3) \qquad g_{ij,p} + g_{jp,i} + g_{pi,j} = 0,$$

where $g_{ij.p}$ is the covariant derivative of g_{ij}. By differentiating (19.3) covariantly, we obtain

$$g_{ij,p,q} + g_{jp,i,q} + g_{pi,j,q} = 0,$$

$$g_{ij,p,q,r} + g_{jp,i,q,r} + g_{pi,j,q,r} = 0,$$

$$g_{ij,p,q,r,s} + g_{jp,i,q,r,s} + g_{pi,j,q,r,s} = 0.$$

By substituting (12.4), (12.5) and (12.6) in these three equations we obtain

$$g_{ij,pq} + g_{jp,iq} + g_{pi,jq} = P_{ijpq}, \tag{19.4}$$

$$g_{ij,pqr} + g_{jp,iqr} + g_{pi.jqr} = P_{ijpqr}, \tag{19.5}$$

$$g_{ij,pqrs} + g_{jp,iqrs} + g_{pi,jqrs} = P_{ijpqrs}, \tag{19.6}$$

where

$$P_{ijpq} = 2(g_{i\alpha} A^{\alpha}_{jpq} + g_{j\alpha} A^{\alpha}_{piq} + g_{p\alpha} A^{\alpha}_{ijq}), \tag{19.7}$$

$$\begin{aligned} P_{ijpqr} = 2(&g_{i\alpha,q} A^{\alpha}_{jpr} + g_{j\alpha,q} A^{\alpha}_{pir} + g_{p\alpha,q} A^{\alpha}_{ijr} \\ &+ g_{i\alpha,r} A^{\alpha}_{jpq} + g_{j\alpha,r} A^{\alpha}_{piq} + g_{p\alpha,r} A^{\alpha}_{ijq} \\ &+ g_{i\alpha} A^{\alpha}_{jpqr} + g_{j\alpha} A^{\alpha}_{piqr} + g_{p\alpha} A^{\alpha}_{ijqr}), \end{aligned} \tag{19.8}$$

$$\begin{aligned} P_{ijpqrs} = 2(&g_{i\alpha,qr} A^{\alpha}_{jps} + g_{i\alpha,rs} A^{\alpha}_{jpq} + g_{i\alpha,sq} A^{\alpha}_{jpr} \\ &+ g_{j\alpha,qr} A^{\alpha}_{ips} + g_{j\alpha,rs} A^{\alpha}_{ipq} + g_{j\alpha,sq} A^{\alpha}_{ipr} \\ &+ g_{p\alpha,qr} A^{\alpha}_{ijs} + g_{p\alpha,rs} A^{\alpha}_{ijq} + g_{p\alpha,sq} A^{\alpha}_{ijr} \\ &+ g_{i\alpha,q} A^{\alpha}_{jprs} + g_{i\alpha,r} A^{\alpha}_{jpqs} + g_{i\alpha,s} A^{\alpha}_{jpqr} \\ &+ g_{j\alpha,q} A^{\alpha}_{iprs} + g_{j\alpha,r} A^{\alpha}_{ipqs} + g_{j\alpha,s} A^{\alpha}_{ipqr} \\ &+ g_{p\alpha,q} A^{\alpha}_{ijrs} + g_{p\alpha,r} A^{\alpha}_{ijqs} + g_{p\alpha,s} A_{ijqr} \\ &+ g_{i\alpha} A^{\alpha}_{jpqrs} + g_{j\alpha} A^{\alpha}_{ipqrs} + g_{p\alpha} A^{\alpha}_{ijqrs}). \end{aligned} \tag{19.9}$$

The P's are tensors which are symmetric in the first three indices and also in the remaining ones. Thus

$$
\begin{aligned}
& P_{ijpq} = P_{jipq} = P_{ipjq}, \\
(19.10) \qquad & P_{ijpqr} = P_{jipqr} = P_{ijprq} \text{ etc.}, \\
& P_{ijpqrs} = P_{jipqrs} = P_{ijpqsr} \text{ etc.}
\end{aligned}
$$

The equations (19.4) can not be solved for $g_{ij,pq}$ but may be solved for the difference of two of these extensions, namely

$$
(19.11) \qquad g_{ij,pq} - g_{pq,ij} = \frac{1}{2}\left(P_{ijpq} + P_{ijqp} - P_{jpqi} - P_{ipqj}\right).
$$

We may however solve* (19.5) for $g_{ij,pqr}$, thus

$$
(19.12) \qquad \begin{aligned}
g_{ij,pqr} = {} & \frac{1}{3}\left(P_{ijpqr} + P_{ijqpr} + P_{ijrpq} + P_{pqrij}\right) \\
& - \frac{1}{6}\left(P_{ipqjr} + P_{irpjq} + P_{iqrjp} + P_{jpqir} + P_{jrpiq} + P_{jqrip}\right).
\end{aligned}
$$

Similarly from (19.6)

$$
(19.13) \qquad \begin{aligned}
g_{ij,pqrs} = {} & \frac{1}{3}\left(P_{ijpqrs} + P_{ijqprs} + P_{ijrpqs} + P_{pqrijs}\right) \\
& - \frac{1}{6}\left(P_{ipqjrs} + P_{irpjqs} + P_{iqrjps} + P_{jpqirs} + P_{jrpiqs} + P_{jqrips}\right).
\end{aligned}
$$

The equations (19.11) and (19.12) constitute integrability conditions arising from second and third derivatives respectively. By interchanging r and s in (19.13) and subtracting we obtain the integrability conditions arising from the fourth derivatives:

$$
(19.14) \qquad \begin{aligned}
& \left(2P_{ijrqps} + 2P_{pqrijs} + P_{ispjqr} + P_{iqsjpr} + P_{jspiqr} + P_{jqsipr}\right) \\
& - \left(2P_{ijsqpr} + 2P_{pqsijr} + P_{irpjqs} + P_{iqrjps} + P_{jrpiqs} + P_{jqrips}\right) = 0.
\end{aligned}
$$

* The solution is facilitated by noticing that the tensors in the left member of (19.5) can be regarded as notation for the vertices of a Desargues configuration.

If we substitute the value of P_{ijpqrs} given by (19.9) in (19.14) we obtain an equation which we may write in the form

$$g_{\alpha\beta}\,U^{\alpha\beta}_{ijpqrs} + g_{\alpha\beta,\gamma}\,V^{\alpha\beta\gamma}_{ijpqrs} + g_{\alpha\beta,\gamma\delta}\,W^{\alpha\beta\gamma\delta}_{ijpqrs} = 0, \tag{19.15}$$

where U, V, and W are tensors.

Let us next consider the identity

$$g_{ij,p,q} = g_{ij,pq} - g_{\alpha j}\,A^{\alpha}_{ipq} - g_{i\alpha}\,A^{\alpha}_{jpq} \tag{19.16}$$

(cf. (12.4)). The third covariant derivative $g_{ij,p,q,r}$ may be evaluated in terms of g_{ij} and $g_{ij,p}$ by setting the value of $g_{ij,pqr}$ given by (19.12) in the identity (12.5):

$$\begin{aligned} g_{ij,p,q,r} = {} & g_{\alpha i,p}\,A^{\alpha}_{jrq} + g_{\alpha i,q}\,A^{\alpha}_{jrp} + g_{\alpha i,r}\,A^{\alpha}_{jqp} + g_{\alpha j,p}\,A^{\alpha}_{riq} + g_{\alpha j,q}\,A^{\alpha}_{rip} \\ & + g_{\alpha j,r}\,A^{\alpha}_{qip} + g_{\alpha p,i}\,A^{\alpha}_{qrj} + g_{\alpha p,j}\,A^{\alpha}_{qri} + g_{\alpha p,q}\,A^{\alpha}_{ijr} + g_{\alpha p,r}\,A^{\alpha}_{ijq} \\ & + g_{\alpha q,i}\,A^{\alpha}_{rpj} + g_{\alpha q,j}\,A^{\alpha}_{rpi} + g_{\alpha q,p}\,A^{\alpha}_{ijr} + g_{\alpha q,r}\,A^{\alpha}_{ijp} + g_{\alpha r,i}\,A^{\alpha}_{pqj} \\ & + g_{\alpha r,j}\,A^{\alpha}_{pqi} + g_{\alpha r,p}\,A^{\alpha}_{ijq} + g_{\alpha r,q}\,A^{\alpha}_{ijp} - g_{ij,\alpha}\,A^{\alpha}_{pqr} \\ & + g_{\alpha i}\,(A^{\alpha}_{jqpr} + A^{\alpha}_{jrpq}) + g_{\alpha j}\,(A^{\alpha}_{qipr} + A^{\alpha}_{riqp}) \\ & + g_{\alpha p}\,(A^{\alpha}_{ijqr} + A^{\alpha}_{qrij}) + g_{\alpha q}\,(A^{\alpha}_{ijpr} + A^{\alpha}_{prij}) \\ & + g_{\alpha r}\,(A^{\alpha}_{ijpq} + A^{\alpha}_{pqij}). \end{aligned} \tag{19.17}$$

If we differentiate both members of (19.16) covariantly and substitute for $g_{ij,p,q,r}$ its value from (19.17), we obtain an equation which we may write as

$$g_{ij,pq,r} = g_{\alpha\beta}\,E^{\alpha\beta}_{ijpqr} + g_{\alpha\beta,\gamma}\,F^{\alpha\beta\gamma}_{ijpqr}, \tag{19.18}$$

where E and F are tensors. Next differentiate (19.15) covariantly, obtaining

$$\begin{aligned} & g_{\alpha\beta,i}\,U^{\alpha\beta} + g_{\alpha\beta}\,U^{\alpha\beta}_{,i} + g_{\alpha\beta,\gamma,i}\,V^{\alpha\beta\gamma} + g_{\alpha\beta,\gamma}\,V^{\alpha\beta\gamma}_{,i} \\ & \qquad + g_{\alpha\beta,\gamma\sigma,i}\,W^{\alpha\beta\gamma\sigma} + g_{\alpha\beta,\gamma\sigma}\,W^{\alpha\beta\gamma\sigma}_{,i} = 0, \end{aligned} \tag{19.19}$$

in which we have omitted the covariant indices for simplicity. When we make the substitutions (19.16) and (19.18) this equation becomes

$$
\begin{aligned}
&g_{\alpha\beta,i}\,U^{\alpha\beta}+g_{\alpha\beta}\,U^{\alpha\beta}_{,i}+(g_{\alpha\beta,\gamma i}-g_{\sigma\beta}\,A^{\sigma}_{\alpha\gamma i}-g_{\alpha\sigma}\,A^{\sigma}_{\beta\gamma i})\,V^{\alpha\beta\gamma}\\
&+g_{\alpha\beta,\gamma}\,V^{\alpha\beta\gamma}_{,i}+(g_{\mu\nu}\,E^{\mu\nu}_{\alpha\beta\gamma\sigma i}+g_{\mu\nu,\eta}\,F^{\mu\nu\eta}_{\alpha\beta\gamma\sigma i})\,W^{\alpha\beta\gamma\sigma}+g_{\alpha\beta,\gamma\sigma}\,W^{\alpha\beta\gamma\sigma}_{,i}=0
\end{aligned} \tag{19.20}
$$

and may be written in the form (19.15) as

$$
g_{\alpha\beta}\,U^{\alpha\beta}_{1}+g_{\alpha\beta,\gamma}\,V^{\alpha\beta\gamma}_{1}+g_{\alpha\beta,\gamma\sigma}\,W^{\alpha\beta\gamma\sigma}_{1}=0. \tag{19.21}
$$

By covariant differentiation of (19.21) and substitution for $g_{ij,p,q}$ and $g_{ij,pq,r}$ from (19.16) and (19.18) we again obtain a system of equations of the form (19.15). Continuing this process we are led to the following sequence of systems of equations. As in the case of the linear first integral we add to this sequence the conditions (19.3) and (19.4) and also the symmetry conditions on the g's for the purpose of convenient reference:

$$
\left\{
\begin{aligned}
&g_{ij,p}+g_{jp,i}+g_{pi,j}=0;\\
&g_{ij,pq}+g_{jp,iq}+g_{pi,jq}=2\,(g_{i\alpha}\,A^{\alpha}_{jpq}+g_{j\alpha}\,A^{\alpha}_{piq}+g_{p\alpha}\,A^{\alpha}_{ijq});\\
&g_{ij}=g_{ji};\quad g_{ij,p}=g_{ji,p};\quad g_{ij,pq}=g_{ji,pq};\quad g_{ij,pq}=g_{ij,qp};
\end{aligned}
\right.
$$

$$
\begin{aligned}
&g_{\alpha\beta}\,U^{\alpha\beta}+g_{\alpha\beta,\gamma}\,V^{\alpha\beta\gamma}+g_{\alpha\beta,\gamma\sigma}\,W^{\alpha\beta\gamma\sigma}=0,\\
&g_{\alpha\beta}\,U^{\alpha\beta}_{1}+g_{\alpha\beta,\gamma}\,V^{\alpha\beta\gamma}_{1}+g_{\alpha\beta,\gamma\sigma}\,W^{\alpha\beta\gamma\sigma}_{1}=0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&g_{\alpha\beta}\,U^{\alpha\beta}_{n}+g_{\alpha\beta,\gamma}\,V^{\alpha\beta\gamma}_{n}+g_{\alpha\beta,\gamma\sigma}\,W^{\alpha\beta\gamma\sigma}_{n}=0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots.
\end{aligned} \tag{19.22}
$$

The algebraic consistency of the equations (19.22) is a necessary condition for the existence of a homogeneous quadratic first integral. As in the preceding cases there must therefore exist a number N such that the first $N+1$ systems of equations of (19.22) possess a fundamental set of s solutions $g^{(\alpha)}_{ij}$; $g^{(\alpha)}_{ijp}$; $g^{(\alpha)}_{ijpq}$ $(\alpha = 1, 2, \dots s)$ each of which satisfies the $(N+2)$nd system of the sequence. We shall show that this is also a sufficient condition for the existence of the quadratic integral (19.1).

We first take the case in which $N = 1$, and $s = 1$. The first two systems of equations (19.22) then possess a unique solution which satisfies (19.20). This solution possesses a tensor character so that we may substitute it in (19.15) and differentiate covariantly, obtaining

$$\begin{aligned} g_{\alpha\beta,i}\, U^{\alpha\beta} + g_{\alpha\beta}\, U^{\alpha\beta}_{,i} + g_{\alpha\beta\gamma,i}\, V^{\alpha\beta\gamma} + g_{\alpha\beta\gamma}\, V^{\alpha\beta\gamma}_{,i} \\ + g_{\alpha\beta\gamma\sigma,i}\, W^{\alpha\beta\gamma\sigma} + g_{\alpha\beta\gamma\sigma}\, W^{\alpha\beta\gamma\sigma}_{,i} = 0. \end{aligned} \tag{19.23}$$

Subtracting (19.20), into which the solution g_{ij}; g_{ijp}; g_{ijpq} has been substituted instead of g_{ij}; $g_{ij,p}$; $g_{ij,pq}$, from (19.23),

$$\begin{aligned} (g_{\alpha\beta,i} - g_{\alpha\beta i})\, U^{\alpha\beta} + (g_{\alpha\beta\gamma,i} - g_{\alpha\beta\gamma i} + g_{\sigma\beta}\, A^{\sigma}_{\alpha\gamma i} + g_{\alpha\sigma}\, A^{\sigma}_{\beta\gamma i})\, V^{\alpha\beta\gamma} \\ + (g_{\alpha\beta\gamma\sigma,i} - g_{\mu\nu}\, E^{\mu\nu}_{\alpha\beta\gamma\sigma i} - g_{\mu\nu\eta}\, F^{\mu\nu\eta}_{\alpha\beta\gamma\sigma i})\, W^{\alpha\beta\gamma\sigma} = 0. \end{aligned} \tag{19.24}$$

The solution appearing in (19.24) satisfies the first system of equations (19.22) in the summed indices and is consequently given by

$$g_{ij,p} - g_{ijp} = \varphi_p\, g_{ij}, \tag{19.25}$$

$$g_{ijp,q} - g_{ijpq} + g_{\alpha j}\, A^{\alpha}_{ipq} + g_{i\alpha}\, A^{\alpha}_{jpq} = \varphi_q\, g_{ijp}, \tag{19.26}$$

$$g_{ijpq,r} - g_{\alpha\beta}\, E^{\alpha\beta}_{ijpqr} - g_{\alpha\beta\gamma}\, F^{\alpha\beta\gamma}_{ijpqr} = \varphi_r\, g_{ijpq}. \tag{19.27}$$

The covariant vector φ_i is the gradient of a scalar function, for if we differentiate (19.25) covariantly,

$$g_{ij,p,q} - g_{ijp,q} = \varphi_{p,q}\, g_{ij} + \varphi_p\, g_{ij,q}.$$

Hence,

$$(\varphi_{p,q} - \varphi_{q,p})\, g_{ij} = g_{ij,p,q} - g_{ij,q,p} + g_{ijq,p} - g_{ijp,q} + \varphi_q\, g_{ij,p} - \varphi_p\, g_{ij,q}. \tag{19.28}$$

If we substitute the values given by (12.4), (19.25), and (19.26) for the covariant derivatives in (19.28) we find that the right member vanishes identically. In the left member of (19.28) g_{ij} can not be equal to zero, for if this were so we see from (19.25) and (19.26) that g_{ijp} and g_{ijpq} would also vanish, which is contrary to the assumption that our equations are algebraically consistent. Hence

$$\varphi_{p,q} - \varphi_{q,p} = 0,$$

or

$$\varphi_p = \frac{\partial \varphi}{\partial x^p}. \tag{19.29}$$

A homogeneous quadratic first integral (19.1) will exist if a function ψ can be chosen so that

$$(\psi g_{ij})_p = \psi g_{ijp}, \tag{19.30}$$

where the left member denotes the covariant derivative of ψg_{ij}. If we expand (19.30) we find that the condition takes the form

$$\frac{\partial \log \psi}{\partial x^p} + \varphi_p = 0. \tag{19.31}$$

Therefore (19.30) is satisfied if we put

$$\psi = e^{-\varphi}.$$

Hence, *a sufficient condition for the existence of a homogeneous quadratic first integral* (19.1) *is that the first two systems of equations* (19.22) *possess a unique solution which satisfies the third system.*

We return now to the general case and assume that there exists a number N such that the first $(N+1)$ systems of equations (19.22) admit a fundamental set of s solutions $g_{ij}^{(\alpha)}$; $g_{ijp}^{(\alpha)}$; $g_{ijpq}^{(\alpha)}$ $(\alpha = 1, 2, \ldots, s)$ each of which satisfies the $(N+2)$nd system of equations. The general solution of the first $(N+1)$ systems of equations may then be written

$$g_{ij} = \varphi^{(\alpha)} g_{ij}^{(\alpha)}, \tag{19.32}$$

$$g_{ijp} = \varphi^{(\alpha)} g_{ijp}^{(\alpha)}, \tag{19.33}$$

$$g_{ijpq} = \varphi^{(\alpha)} g_{ijpq}^{(\alpha)}, \tag{19.34}$$

where the right members are summed for α from $\alpha = 1$ to $\alpha = s$. Let us substitute the particular solution $g_{ij}^{(k)}$; $g_{ijp}^{(k)}$; $g_{ijpq}^{(k)}$ in the equations (19.22) beginning with the second system and ending with the $(N+2)$nd. If we then differentiate each system of equations through the $(N+1)$st covariantly, and subtract it from the system immediately following, we shall obtain the equations

$$
\begin{aligned}
&(\overset{(k)}{g}_{\alpha\beta,i} - \overset{(k)}{g}_{\alpha\beta i})\, U^{\alpha\beta} + (\overset{(k)}{g}_{\alpha\beta\gamma,i} - \overset{(k)}{g}_{\alpha\beta\gamma i} + \overset{(k)}{g}_{\sigma\beta}\, A^{\sigma}_{\alpha\gamma i} + \overset{(k)}{g}_{\alpha\sigma}\, A^{\sigma}_{\beta\gamma i})\, V^{\alpha\beta\gamma} \\
&\qquad + (\overset{(k)}{g}_{\alpha\beta\gamma\sigma,i} - \overset{(k)}{g}_{\mu\nu}\, E^{\mu\nu}_{\alpha\beta\gamma\sigma i} - \overset{(k)}{g}_{\mu\nu\eta}\, F^{\mu\nu\eta}_{\alpha\beta\gamma\sigma i})\, W^{\alpha\beta\gamma\sigma} = 0, \\
&(\overset{(k)}{g}_{\alpha\beta,i} - \overset{(k)}{g}_{\alpha\beta i})\, U_1^{\alpha\beta} + (\overset{(k)}{g}_{\alpha\beta\gamma,i} - \overset{(k)}{g}_{\alpha\beta\gamma i} + \overset{(k)}{g}_{\sigma\beta}\, A^{\sigma}_{\alpha\gamma i} + \overset{(k)}{g}_{\alpha\sigma}\, A^{\sigma}_{\beta\gamma i})\, V_1^{\alpha\beta\gamma} \\
&\qquad + (\overset{(k)}{g}_{\alpha\beta\gamma\sigma,i} - \overset{(k)}{g}_{\mu\nu}\, E^{\mu\nu}_{\alpha\beta\gamma\sigma i} - \overset{(k)}{g}_{\mu\nu\eta}\, F^{\mu\nu\eta}_{\alpha\beta\gamma\sigma i})\, W_1^{\alpha\beta\gamma\sigma} = 0, \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&(\overset{(k)}{g}_{\alpha\beta,i} - \overset{(k)}{g}_{\alpha\beta i})\, U_{N-1}^{\alpha\beta} + (\overset{(k)}{g}_{\alpha\beta\gamma,i} - \overset{(k)}{g}_{\alpha\beta\gamma i} + \overset{(k)}{g}_{\sigma\beta}\, A^{\sigma}_{\alpha\gamma i} + \overset{(k)}{g}_{\alpha\sigma}\, A^{\sigma}_{\beta\alpha i})\, V_{N-1}^{\alpha\beta\gamma} \\
&\qquad + (\overset{(k)}{g}_{\alpha\beta\gamma\sigma,i} - \overset{(k)}{g}_{\mu\nu}\, E^{\mu\nu}_{\alpha\beta\gamma\sigma i} - \overset{(k)}{g}_{\mu\nu\eta}\, F^{\mu\nu\eta}_{\alpha\beta\gamma\sigma i})\, E_{N-1}^{\alpha\beta\gamma\sigma} = 0.
\end{aligned}
\tag{19.35}
$$

Since the solution which appears in (19.35) satisfies the first $(N+1)$ systems of equations (19.22), it may be written as a linear combination of the fundamental solutions $g_{ij}^{(k)}$; $g_{ijp}^{(k)}$; $g_{ijpq}^{(k)}$, $k = 1, 2, \ldots . s$. Hence

$$g_{ij,p}^{(k)} - g_{ijp}^{(k)} = \lambda_p^{(k\alpha)} g_{ij}^{(\alpha)}, \tag{19.36}$$

$$g_{ijp,q}^{(k)} - g_{ijpq}^{(k)} + g_{\sigma j}^{(k)} A_{ipq}^{\sigma} + g_{i\sigma}^{(k)} A_{jpq}^{\sigma} = \lambda_q^{(k\alpha)} g_{ijp}^{(\alpha)}, \tag{19.37}$$

$$g_{ijpq,r}^{(k)} - g_{\mu\nu}^{(k)} E_{ijpqr}^{\mu\nu} - g_{\mu\nu\eta}^{(k)} F_{ijpqr}^{\mu\nu\eta} = \lambda_r^{(k\alpha)} g_{ijpq}^{(\alpha)}, \tag{19.38}$$

the left members of these equations being summed for α from $\alpha = 1$ to $\alpha = s$. To find the conditions which the covariant vectors λ must satisfy we proceed in the same way as for the linear integral and thus obtain the equations

$$\begin{aligned} g_{ij}^{(\beta)} \left(\lambda_{p,q}^{(\alpha\beta)} - \lambda_{q,p}^{(\alpha\beta)} + \lambda_p^{(\alpha\gamma)} \lambda_q^{(\gamma\beta)} - \lambda_q^{(\alpha\gamma)} \lambda_p^{(\gamma\beta)}\right) &= 0, \\ g_{ijp}^{(\beta)} \left(\lambda_{q,r}^{(\alpha\beta)} - \lambda_{r,q}^{(\alpha\beta)} + \lambda_q^{(\alpha\gamma)} \lambda_r^{(\gamma\beta)} - \lambda_r^{(\alpha\gamma)} \lambda_q^{(\gamma\beta)}\right) &= 0, \\ g_{ijpq}^{(\beta)} \left(\lambda_{r,s}^{(\alpha\beta)} - \lambda_{s,r}^{(\alpha\beta)} + \lambda_r^{(\alpha\gamma)} \lambda_s^{(\gamma\beta)} - \lambda_s^{(\alpha\gamma)} \lambda_r^{(\gamma\beta)}\right) &= 0, \end{aligned} \tag{19.39}$$

which are summed for β from $\beta = 1$ to $\beta = s$. It follows consequently that

$$\lambda_{p,q}^{(\alpha\beta)} - \lambda_{q,p}^{(\alpha\beta)} + \lambda_p^{(\alpha\gamma)} \lambda_q^{(\gamma\beta)} - \lambda_q^{(\alpha\gamma)} \lambda_p^{(\gamma\beta)} = 0, \tag{19.40}$$

for otherwise one of the fundamental solutions could be expressed linearly in terms of the others.

A homogeneous quadratic first integral (19.1) will exist provided that the arbitrary functions φ in the general solution can be so chosen that

$$(\varphi^{(\alpha)} g_{ij}^{(\alpha)})_p = \varphi^{(\alpha)} g_{ijp}^{(\alpha)}. \tag{19.41}$$

where the term on the left denotes the covariant derivative of $\varphi^{(\alpha)} g_{ij}^{(\alpha)}$. If we expand (19.41) we find that the quadratic integral will exist if the φ's can be chosen so as to satisfy the equation

$$\frac{\partial \varphi^{(\beta)}}{\partial x^p} + \varphi^{(\alpha)} \lambda_p^{(\alpha\beta)} = 0. \tag{19.42}$$

The integrability condition of (19.42) is the equation (19.40) so that a set of φ's can be found which will satisfy (19.42).

Hence, *a necessary and sufficient condition for the existence of a quadratic first integral* (19.1) *is that the Γ's be such that there exists a number N such that the first* $(N+1)$ *systems of equations* (19.22) *admit a fundamental set of s solutions* $(s \geqq 1)$ *which satisfies the* $(N+2)$*nd system of equations.*

PRINCETON UNIVERSITY,
PRINCETON, N. J.